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Note: These solutions are a work in progress; comments, references, etc. are appreciated. Most references and figures can be found with the problem statements.

Problem 1. At each move a player at roulette bets on any of 38 equiprobable outcomes; if his outcome occurs he wins \$36, and otherwise he loses \$1. Let p_n denote the probability that the house is ahead after n moves. Which of the following holds?

(a) $0.99 < p_{105}$

(b) $0.95 < p_{105} < 0.99$

(c) $0.90 < p_{105} < 0.95$

(d) $0.80 < p_{105} < 0.90$

(e) $0.70 < p_{105} < 0.80$

(f) $0.60 < p_{105} < 0.70$

(g) $0.50 < p_{105} < 0.60$

(h) $p_{105} < 0.50$

Discussion: With $n = 105$, the better loses if his outcome comes up 0, 1, or 2 times, and wins if it comes up 3 or more times. The probability 0, 1, or 2 successes in $n = 105$ spins, each with probability $p = 1/38$ of success is

$$(1-p)^n + \binom{n}{1}(1-p)^{n-1}p + \binom{n}{2}(1-p)^{n-2}p^2.$$

(E.g., the last term is the number of ways to choose 2 out of the n spins times the probability that those 2 will be successful and the remainder not.) A straightforward numerical calculation shows that this is slightly less than .476, so that the probability that the bettor is ahead after 105 rolls is more than .524, i.e., answer (h) is correct.

It is perhaps more interesting to see how one could guess this by a back-of-the-envelope calculation. The Poisson approximation comes to our rescue. Let $m = 36$, and suppose that we alter p and n slightly to $p = 1/m$, $n = 108 = 3m$. Then the probability that the better is behind (i.e., strictly less than 3 successes) becomes

$$\left(1-\frac{1}{m}\right)^{3m} + 3m\left(1-\frac{1}{m}\right)^{3m-1}\frac{1}{m} + \frac{3m(3m-1)}{2}\left(1-\frac{1}{m}\right)^{3m-2}\frac{1}{m^2}.$$

The approximation $(1-1/x)^{ax} \simeq e^a$ (for large x) shows that this is approximated by

$$(1+3+9/2)e^{-3} \simeq \frac{8.5}{20} = 42.5\%$$

(remembering the convenient fact that $e^3 = 20.0855\ldots \simeq 20$). This can easily be done on the back of an envelope, and would no doubt lead us to guess (h).

One of the author's sons guessed (h) straightaway merely on the grounds of gamesmanship.

Problem 2. The number 2^{29} is a 9-digit number with distinct digits. Which digit is missing?

Discussion: Since $2^6 \equiv 1 \bmod 9$ it is easy to work out that $2^{29} \equiv 5 \bmod 9$. The sum of all 10 digits is divisible by 9, so casting out nines (i.e., finding 2^29 modulo 9) gives $9 - d$ where d is the missing digit. Therefore, the missing digit is a 4. In fact, $2^{29} = 536870912$.

Problem 3. In a small town there are five families, having 1, 2, 3, 5, and 9 children respectively. The average family size is 4 or, as we might say slightly more colloquially, the average family has 4 children. On the other hand, the average child has 5 siblings, i.e., the average child is in a family of 6 children. Show that in any town the family size of an average child is equal to the size of an average family if and only if all families are the same size.

Discussion: If there are n families of sizes $x_1, x_2, \ldots$ and there are $c := \sum x_i$ children, then the average family size is c/n. The average number of siblings per child is $\sum x_i(x_i - 1)/c$ and each child is in a family of size

$$\frac{\sum x_i(x_i - 1)}{c} + 1 = \frac{\sum x_i^2}{c}$$

and the desired inequality reduces to the well-known fact that

$$\left(\sum x_i\right)^2 \leq n \sum x_i^2)$$

with equality only if all of the x_i are equal. (This is the Cauchy-Schwarz inequality for the vectors $u = (1, 1, \ldots, 1)$ and $v = (x_1, x_2, \ldots, x_n)$.)

Problem 4. If n is a positive integer, let $\mathrm{Odd}(n)$ be $n/2^k$, where 2^k is the largest power of 2 that divides n. Given two positive integers a, b define a sequence by

$$x_1 = a, \qquad x_2 = b, \qquad x_{n+1} = \mathrm{Odd}(x_n + x_{n-1}).$$

For which a, b does the limit of the sequence exist? Can you describe the limit?

Discussion: From the x_3 on, all terms are odd. If any two consecutive terms are equal, the sequence stabilizes (has a limit). Otherwise, the quantity $\max(x_{n-1}, x_n)$ is strictly decreasing since

$$\mathrm{Odd}(x_{n-1} + x_n) < \frac{x_{n-1} + x_n}{2} \leq \max(x_{n-1}, x_n).$$

An infinite strictly decreasing sequence of positive integers is impossible, so the limit exists.

Let $L = L(a, b)$ denote the limit. If g is any odd divisor of both a and b then g divides every term of the sequence and hence divides L. On the other hand, if L divides both x_{n+1} and x_n then the equation

$$x_{n-1} + x_n = 2^k x_{n+1}$$

implies that L divides x_{n-1}. Working backwards, we see that L divides both a and b, and we conclude that $L = \gcd(\mathrm{Odd}(a), \mathrm{Odd}(b))$.

Problem 5. A (long) line of lockers are labeled $1, 2, \ldots$, and they are opened/closed by people labeled $1, 2, \ldots$ as follows: For each k, person k walks down the line, reversing the open/closed state of each locker with label divisible by k (i.e., opening a closed locker, and shutting an open locker). A well-known puzzle asks which lockers are open after everyone has walked down the line; the answer is that the lockers whose labels are perfect squares are open (proof left to the reader).

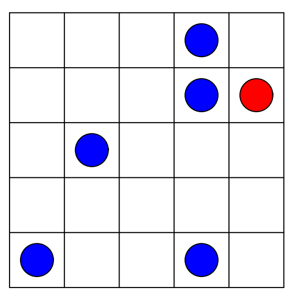
Describe a set S of positive integers such that if, for each k in S, person k is sent down the line then only the first locker is open afterwards.

Discussion: (Everyone walks down the line.) The state of locker m is changed once for every divisor of m. The number of divisors of $m = \prod p_i^{e_i}$ is $\prod e_i + 1$, where the p_i are the distinct prime divisors of m. This is odd (i.e., the locker is left open) if and only if every e_i is even, i.e., m is a perfect square. Thus the lockers left open after everyone has gone down the line are equality those with a square number.

(Leaving only the first locker open.) A similar analysis shows that if the people k that walk down the line are exactly those with square-free k (i.e., those for which k is the product of distinct primes) then only the first locker is left open. Indeed, if $m = \prod p_i^{e_i}$ then the square-free divisors of m consist of products of the distinct p_i that divide m, of which there are 2^N, where N is the number of prime divisors of m. This is even if and only if $N = 0$, i.e., $m = 1$. Thus the only locker left open is the very first one.

Note that a simple induction argument shows that if S is a set of positive integers, then there exists some set T of positive integers such that if $k \in T$ walk down the line the result is that the lockers open are exactly those with numbers in S. (After deciding whether to send person k down the line for $k < n$, decide whether to send person n based on the current state of the locker.)

Problem 6. On the 5×5 board below, any of the 6 red or blue pieces can jump over one or more adjacent pieces and land on the next open space. (Jumped pieces are not removed from the board, and diagonal jumps are not allowed.) Find a sequence of moves that ends with the red piece on the lower right hand square.



Discussion: Six immortal pieces are located in a 5x5 board. These six pieces are denoted by the letters `K` or `s`; the initially empty squares are denoted by `0`:

```
0s000
Ks000
000s0
00000
0s00s
```

A legal move consists of taking any piece and jumping in a straight horizontal or vertical line over a consecutive string of one or more immediately adjacent pieces, and landing on the first empty square thereafter. One of the pieces, marked `K`, is called a king even though its legal moves are no different from any of the other pieces. `K` starts on the square shown. The problem is to get this piece onto the lower left corner square in a finite number of legal moves.

Label the squares of the board as follows:

```
0C0C0
ABABA
0C0C0
0B0B0
DCDCD
```

Then notice that the set of letters initially occupied by the 6 pieces is A,B,C,C,C,D. Further notice that any single or triple jump necessarily leaves this set of letters unchanged. Further notice that any double jump can only change A to B or B to C or C to D, or, in reverse, from D to C, or C to B, or B to A. It is now clear that the squares marked 0 in this latter diagram must always remain vacant.

It is convenient to partition all possible positions into equivalence classes, each of which is characterized by the set of letters which are occupied. Then single jumps move only between positions within the same equivalence class, whereas double jumps change classes.

We also notice that in any double jump, the jumpees must have the same pair of letters as the origin and destination of the jumper. This implies that the set of letters occupied by the pieces must always include at least one of each of the four letters. So we can simplify our notation to indicate only the surplus letters. When the king does not occupy a surplus letter, we use a pair of small letters to characterize the position. But when the king is on a surplus letter, we will capitalize that letter. The initial position is cc. The final position must be Dx, for some appropriate value of the letter x. The penultimate position must be Cx. Prior positions must evidently include Bx and Ax (for possibly different values of x). The first transition from Ax to Bx will have to be a horizontal king jump. The jumpee which then occupies the A in the central column must have initially gotten to an A square by a double jump from a B square. The king must have been one of its jumpees.

This logical reasoning with equivalence classes leads to a strategic plan: We must transform the surplus c's to one or more surplus b's, one of which must then do a double jump over the king, which must be located on the central column in the AB row. We'll then need to "shuffle" both of the occupants of A squares. The king will have be moved to a side A, and the occupant of the other side A will then have to be moved to the center. So we need to plan where the other four pieces will have to be to accomplish this shuffle. Evidently, one will have to be stationed on the central D square, while the remaining three move about among 2 B's and one C. So all of the shuffling will occur with single jumps in an equivalence class of type Ab.

So the positions in the solution are then found to be the following:

$$\mathtt{cc}^3, \mathtt{bc}^7, \mathtt{bb}^3, \mathtt{Ab}^{14}, \mathtt{Bb}^3, \mathtt{Cb}^3, \mathtt{Db}^1$$

The 18 positions in the central shuffle consist of three sets of 6. Each of these sets begins and ends with a horizontal jump in the AB row.

Problem 7. Identical billiard balls are placed at the vertices of a regular n-gon in an infinite plane. For which values of n is it possible to strike the a ball so that it hits the next ball, which hits the next ball, etc., so that finally the last ball hits the first ball?

We assume that all balls are identical small discs in the plane, where the size of each disc is much smaller than the side of the n-gon. Also note that the n-th ball is required to catch up with the first ball, as the latter is heading outwards to infinity.

Finally, for very large n, how many collisions are possible? (i.e., after the last ball hits the first ball, the first ball hits the second ball, etc.)

Discussion: Before solving the billiards problems, it is convenient to review basic facts about the collision of ideal point billiards. If a ball with velocity vector u strikes a stationary ball and the two balls leave the point of collision with velocity vectors v and w, then conservation of momentum ($u = v + w$) and conservation of kinetic energy ($|u|^2 = |v|^2 + |w|^2$) imply that v and w are perpendicular, and that, e.g., the magnitudes of u and v are related by $|v| = |u|\cos(\theta_{uv})$ where θ_{uv} is the angle between u and v.

Let $p_0, p_1, \ldots, p_{n-1}$ be n points in the plane at the vertices of a regular n-gon, e.g., the n points $p_j = \exp(2\pi i j/n)$ in the complex plane. If $n > 4$ then each ball can hit the next ball all the way around the circle. At each collision the speed is decreased by a factor of $C := \cos(\theta)$, where $\theta = 2\pi i/n$. If the ball at p_0 is initially sent towards p_1 with unit velocity, then p_{n-2} moves towards p_{n-1} with velocity C^{n-2}.

Can the ball at p_{n-1} be struck so as to hit the original ball? The original ball is now traveling towards infinity in a direction perpendicular to the line segment $p_1 p_2$ with speed $S := \sin(\theta)$; let v denote that velocity vector. The ball at p_{n-1} will be hit by a ball traveling at velocity C^{n-2} in the direction of the line segment $p_{n-2}p_{n-1}$; let u denote that velocity vector. The struck ball can overtake and collide with the original ball if and only if the projection of that velocity vector onto the vector v has magnitude greater than the magnitude of v; this is equivalent to saying that $\cos(\theta_{uv} C^{n-2} > S$. Since the angle between u and v is $3\theta - \pi/2$ this is equivalent to

$$C^{n-2}\cos(3\theta - \pi/2) > S$$

which (after some trigonometric juggling) reduces to $4C^n > C^{n-2} + 1$. A numerical investigation shows that the smallest n for which this is (barely) true is $n = 19$. In other words, a (truly) perfect billiards player can cause the first ball can be overtaken by the last ball if and only if $n \geq 19$.

The second problem requires a similar, and more elaborate, analysis. The brief summary is as follows: The first $n-2$ balls travel outwards at a velocity of $O(1/n)$. The elapsed time until ball $n-1$ hits ball n is $O(1)$. To within $O(1/n)$ the velocities of ball 1 and ball $n-2$ are perpendicular. Ball 1 is 2 units ahead of ball $n-1$ and 2π units to the right (to within $O(1/n)$). Ball 1 can catch up with ball 2, ball 2 can catch up to ball 3, etc., all the way up to ball $n-1$. At each collision, a factor of $1/\sqrt{1+\pi^2}$ is lost in speed. This continues until ball $n-2$ is required to hit ball $n-1$. Unfortunately, ball $n-1$ is heading outward with a velocity that exceeds the velocity of any other ball. Thus, for very large n, the perfect billiard player can achieve $2n-2$ consecutive ball to ball collisions, but no more.