

SÉMINAIRE DUBREIL.

ALGÈBRE

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Séminaire Dubreil. Algèbre, tome 28, n° 1 (1974-1975), exp. n° 20, p. 1-4.

http://www.numdam.org/item?id=SD_1974-1975__28_1_A14_0

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NOTES ON AN EXTENSION OF KRULL'S PRINCIPAL IDEAL THEOREM

by David EISENBUD (*)

In this note, we will propose a generalization of Krull's principal ideal theorem which we can prove to be correct, for example, for rings containing a field. We will then sketch an application to the theory of determinantal ideals ; roughly speaking, our theorem implies a more precise version of the theorems of MACAULAY and EAGON on the heights of determinantal ideals. We also mention a speculative connection between our conjecture and the intersection conjectures of SERRE and PESKINE-SZPIRO. Details will appear elsewhere.

1. The generalized principal ideal theorem.

Throughout this paper, all rings will be assumed commutative and noetherian.

Krull's principal ideal theorem [5] states that an element a in the maximal ideal of a local ring R generates an ideal of height at most one (the apparently sharper statement that the minimal primes all have height at most one follows trivially by localization). Regarding a as a homomorphism $R \rightarrow R$, and noting that the rank of R , as an R -module, is 1, one might be lead to conjecture that something "similar" can be said about homomorphisms from an arbitrary module into R . To be more precise, one needs first the right notion of the rank of a module. Since we wish to work with homomorphisms to the ring, it is not unreasonable to require that an R -module M should have rank 0 if, and only if, $M^* = \text{Hom}(M, R) = 0$. As with all notions of rank, a module M should have rank $\leq k$ if, and only if, its $(k + 1)^{\text{th}}$ -exterior power has rank 0. These conditions uniquely specify a notion of the rank of a module, which can be more simply put as follows :

Definition. - Let U be the set of nonzerodivisors of R . The rank of a finitely generated R -module M is the minimal number of generators of M_U as an R -module.

Of course, if R is a domain, this is the usual notion of rank. We can now state our conjecture :

Generalized principal ideal conjecture. - Let R be a local ring with maximal ideal J , and let M be a finitely generated R -module of rank n . Let

$$M^* = \text{Hom}(M, R),$$

and let φ be an element of JM^* . Then the height of the ideal $\varphi(M)$ is at most n .

(*) This is a jointwork of E. Graham EVANS, Jr, and myself. Both authors are grateful to the Sloan Foundation and the N.S.F. for partial support.

It is easy to see that, if $M = R^n$, this conjecture becomes the version of the Krull principal ideal theorem which states that the height of a proper n -generator ideal is at most n .

We can prove a weakened form of the conjecture, in which "height" is replaced by "depth", and we can prove the conjecture itself in many cases.

Recall that if $I \subseteq R$ is an ideal, and N is a finitely generated R -module, then $\text{depth}(I, N)$ is the length of a maximal N -sequence in I .

THEOREM 1. - Let R be a noetherian local ring with maximal ideal J . Let M be a finitely generated R -module, and let $\varphi \in JM^*$.

- (a) If N is a finitely generated R -module, then $\text{depth}(\varphi(M), N) \leq \text{rank } M$.
 (b) If R has (possibly not finitely generated) Cohen-Macaulay modules in the sense of HOCHSTER [4], then

$$\text{height } \varphi(M) \leq \text{rank } M .$$

The extra hypothesis of (b) is known to be fulfilled if R contains a field, and is conjectured to be true in general [4].

It is perhaps amusing to note that our conjecture can be reformulated as giving a condition, in terms of the punctured spectrum, for an element of a module to be part of a minimal system of generators :

Conjecture (second version). - Let (R, J) be a local ring of dimension d , and let M be a module of rank $< d$. Suppose that a is an element of M such that, for every prime ideal $P \neq J$, a generates a free summand of M_P . Then a is part of a minimal system of generators for M .

2. Determinantal ideals.

One of the earliest generalizations of the principal ideal theorem was the theorem of MACAULAY [6] that (for polynomial rings) the height of the ideal of $p \times p$ minors of a $p \times q$ matrix, if the ideal is proper, is at most $q - p + 1$. This was generalized by EAGON in 1960, who showed (for a general noetherian ring) that the height of the ideal of $k \times k$ minors of a $p \times q$ matrix, if the ideal is proper, is at most $(p - k + 1)(q - k + 1)$ (There is a very elegant proof of this in [3]). On the basis of the conjecture made in the last section, we can extend this result to say something about what happens to the ideal of $k \times k$ minors when an extra column is added to the matrix.

Before stating our result, we remark on a result that can be proved by the technique of [3].

PROPOSITION. - Let φ be a $p \times q$ matrix over a ring R , and suppose that the $(\ell + 1) \times (\ell + 1)$ minors of φ are all 0. Then the ideal generated by the $\ell \times \ell$

minors of φ has height at most

$$p + q - 2\ell + 1 .$$

Now suppose that R is local and that we adjoin a new column, with entries in the maximal ideal, to a $p \times q$ matrix φ , obtaining a $p \times (q + 1)$ matrix φ' . Suppose that the $k \times k$ minors of φ are all 0. What can the height h of the ideal of $k \times k$ minors of φ' be? Of course, it is contained in the ideal of $(k - 1) \times (k - 1)$ minors of φ , and also in the ideal generated by the p entries of the new column, so one obtains a bound from the proposition:

$$h \leq \min(p, p + q - 2k + 3) .$$

The next theorem shows that one can do better (at least much of the time!):

THEOREM. - Suppose that R is a local ring satisfying the generalized principal ideal conjecture. (For example, suppose that R contains a field.) Let φ be a $p \times q$ matrix over R whose $k \times k$ minors are all 0, and let φ' be a matrix obtained from φ by adjoining a column whose entries are in the maximal ideal. Then the height of the ideal of $k \times k$ minors of φ' is at most $p - k + 1$.

As a consequence of the theorem and the proposition, we can prove a result which generalizes a "rigidity" theorem of BUCHSBAUM and RIM [1], which, in turn, generalized the result that if n elements f_1, \dots, f_n of a local ring generate an ideal of height n , then any k of them generate an ideal of height k :

COROLLARY. - Suppose that R is a local ring satisfying the generalized principal ideal conjecture, and that φ is a $p \times q$ matrix over R , with coefficients in the maximal ideal, such that the ideal of $k \times k$ minors of φ has height $(p - k + 1)(q - k + 1)$, the largest possible value. Then for every integer $\ell \geq k$, and every $s \times t$ submatrix $\bar{\varphi}$ of φ , the height of the ideal of $\ell \times \ell$ minors of $\bar{\varphi}$ is

$$(s - \ell + 1)(t - \ell + 1) ,$$

again the largest possible value.

In particular, no $\ell \times \ell$ minor of φ is 0.

To see that the hypothesis about coefficients being in the maximal ideal are necessary for this, consider the following matrix over $F[x, y]$, when F is a field:

$$\begin{pmatrix} 0 & 0 & 1 \\ x & y & 0 \end{pmatrix} .$$

Here the height of the ideal of 2×2 minors is 2 ($= (3 - 2 + 1)$), but the first 2×2 minor is 0.

3. A remark on intersection theory.

The remark is easy and speculative : Suppose that M and N are modules of ranks m and n over a local ring (R, J) (containing a field, say). Suppose that $\varphi \in JM^*$ and $\psi \in JN^*$, and write

$$\begin{aligned} X &= \varphi(M) \\ Y &= \psi(N) . \end{aligned}$$

Then the ideal $X + Y$ can be written as $(\varphi, \psi)(M \oplus N)$, so $X + Y$ has height at most $m + n$. This gives some hold on the "intersection theory" of ideals of the form $\varphi(M)$. For example, if one could prove that for every prime ideal P of a regular local ring R, J there exists a module M with $\text{rank } M = \text{ht } P$ and an element $\varphi \in JM^*$ with $\varphi(M) = P$, then one could deduce Serre's intersection theorem ([7], ch. V, theorem 3).

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(Texte reçu le 20 mai 1975)

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