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TRANSCANONICAL EMBEDDINGS OF HYPERELLIPTIC CURVES

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Throughout this paper, a *curve* will be an irreducible nonsingular one-dimensional projective variety over an algebraically closed field F of characteristic not 2.

In Chapter 1 of his beautiful survey [7], Mumford undertakes to exhibit "every curve". The nonhyperelliptic curves are described by the equations of their canonical embeddings [8, 9], but the hyperelliptic curves are given only as ramified double coverings of $\mathbb{P}^1$. In this note we describe the equations and geometry of the "transcanonical" embeddings of hyperelliptic curves, completing, in a sense, the above exhibition.

I would like to thank Sebastian Xambò, who contributed a number of ideas to the formulations below.

We fix for the remainder of the paper a hyperelliptic curve C of genus $g \geq 2$, and let K_0 be the divisor corresponding to the unique map σ of degree 2 from C to the projective line $\mathbb{P}^1$ [5, IV 5.3]. From the Riemann–Roch formula and the criterion [5, IV 3.1] one sees easily that the complete linear series $|(g+k)K_0|$ is very ample iff $k \geq 1$. We call the divisor $(g+k)K_0$, for $k \geq 1$, the kth transcanonical divisor (since the canonical divisor is $(g-1)K_0$), and we write C_k for the image of C under the corresponding embedding in projective space; thus C_k is a nondegenerate (that is, not contained in a hyperplane) curve of degree $2g+2k$ in $\mathbb{P}^{g+2k}$.

We will be particularly concerned with the case $k=1$, which is the simplest. We call $(g+1)K_0$ the hypercanonical divisor, and the corresponding embedding $C \to C_1 \subset \mathbb{P}^{g+2}$ the *hypercanonical* embedding. As we shall see, the hypercanonical divisor could have been defined as the ramification divisor of the canonical map, and thus is well defined even if the base field is not algebraically closed, or even more generally. See [10] for considerations of this type.

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1. Rational normal curves and the hypercanonical embedding

We first recall the basic facts about rational normal curves:
For any integer $n \geq 1$, the image R_n of $\mathbb{P}^1$ in $\mathbb{P}^n$ under the "n-uple embedding"

$$\mathbb{P}^1 \ni (s,\ t) \to (s^n, s^{n-1}t, \ldots, st^{n-1}, t^n)$$

is called the rational normal curve of degree n. In suitable coordinates X_i, the ideal of forms vanishing on R_n may be written as

$$I_{R_n} = \det\begin{pmatrix} X_0 & X_1 & \cdots & X_{n-1} \\ X_1 & X_2 & \cdots & X_n \end{pmatrix};$$

that is, the ideal is generated by the 2×2 minors of the indicated matrix. (We will sketch a proof of this later).

We now return to our hyperelliptic curve C and describe the hypercanonical embedding $C_1 \subset \mathbb{P}^{g+2}$:

Theorem 1. *The $2g+2$ Weierstrass points on C_1 span a hyperplane $W \subset \mathbb{P}^{g+2}$ and lie on a unique rational normal curve $R_{g+1} \subset W$ of degree $g+1$. The tangent lines to C_1 at these points meet in a common point $v \in \mathbb{P}^{g+2} - W$. The curve C_1 is the complete intersection of the cone S with vertex v and base R_{g+1}, with a quadric hypersurface V not passing through v, and σ is the projection $C \to R_{g+1} \cong \mathbb{P}^1$ from v. In suitable coordinates $X_0, \ldots, X_{g+2}$ with $W = \{X_{g+2} = 0\}$ and $v = (0, \ldots, 0, 1)$, the ideal of forms vanishing on C_1 may be written as*

$$I = \left(\det\begin{pmatrix} X_0, X_1, \ldots, X_g \\ X_1, X_2, \ldots, X_{g+1} \end{pmatrix}, Q\right)$$

where $Q = X_{g+2}^2 + Q_1(X_0, \ldots, X_{g+1})$ is a quadratic form.

Conversely, if V is any quadric hypersurface meeting S transversely, then $V \cap S$ is a hypercanonically embedded hyperelliptic curve.

Remarks. (1) The case $g=1$ is not properly included in our Theorem; but if, on a given elliptic curve C, we choose two distinct points (any two such pairs are linearly equivalent after an automorphism of C) and call their sum K_0, then K_0 again determines a map $\sigma : C \to \mathbb{P}^1$ of degree 2, and, defining the Weierstrass points to be the 4 ramification points of this map, the Theorem goes through. This has the advantage that something suggesting the case $g=1$ can be drawn; of course the hypercanonical image C_1 will be the Elliptic normal curve in $\mathbb{P}^3$, the intersection of two quadrics (see Fig. 1).

(2) It follows from the form of the equations given that C_1 is arithmetically normal (that is, its homogeneous coordinate ring is normal). By Serre's Criterion [6, Theorem 39], it is enough to check arithmetic Cohen–Macaulayness, and this is done below.

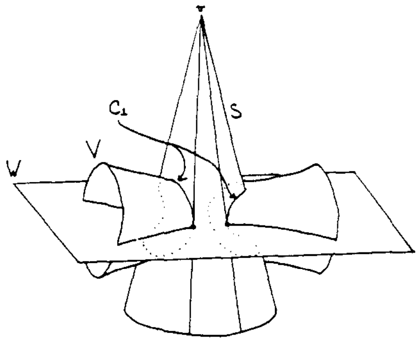

Fig. 1.

Proof. Here is how one might begin synthetically. Let K_{0s}, for $s \in \mathbb{P}^1$, be the family of divisors on C_1 linearly equivalent to K_0. Each K_{0s} consists of two points, and we write L_s for their linear span. Thus $S = \bigcup_s L_s$ is a nondegenerate surface; using, for instance, the classification of surfaces of degree $g+1$ in $\mathbb{P}^{g+2}$, one can show that all the lines L_s meet in a common point v, and thus S is the cone over a rational normal curve.

Instead, we will follow an algebraic line. We may write $u^2 = \prod_1^{2g+2}(t-t_i)$, with distinct t_i, for the affine equation of C. Taking K_0 to be the sum of the two (distinct) points at ∞, we see that $(1, t, t^2, \ldots, t^{g+1}, u)$ is a basis for the space of rational functions with polar divisor $\leq (g+1)K_0$.

Taking $(1, t, t^2, \ldots, t^{g+1}, u)$ to be the coordinates $X_0, \ldots, X_{g+2}$ of points on C_1 in $\mathbb{P}^{g+2}$, we see that

(1) The minors of the matrix

$$\begin{pmatrix} 1 & t & \cdots & t^g \\ t & t^2 & \cdots & t^{g+1} \end{pmatrix}$$

vanish, so C_1 lies on the cone S over an R_{g+1}.

(2) The Weierstrass points of C are given by $u=0$, $t=t_i$; so on C_1 they lie on an R_{g+1} in the hyperplane $W = \{X_{g+2} = 0\}$.

(3) The equation $u^2 = \prod(t-t_i)$ may be written as a quadratic form $Q(X_0 \cdots X_{g+2}) = X_{g+2}^2 + Q_1(X_0, \ldots, X_{g+1})$ in the coordinates, so the ideal I_{C_1} of forms vanishing on C_1 contains an ideal I as described in the Theorem, and setting $V = \{Q=0\}$, we see that $C \subset V \cap S$, and that the Weierstrass points are contained in $V \cap S \cap W$.

We now wish to prove that $I = I_{C_1}$. For this, note that $V \cap S$ is a variety of the same degree $2g+2$ as C_1; thus it suffices to show that $(I_{R_{g+1}}, Q)$ is an unmixed ideal of $F[X_0, \ldots, X_{g+2}]$.

To this end, note first that $F[X_0, \ldots, X_{g+2}]/I_{R_{g+1}}$ is a Cohen–Macaulay domain; Cohen–Macaulay because $I_{R_{g+1}}$ has height g, the generic value for an ideal of minors of a $2 \times (g+1)$ matrix, and such ideals of minors are generically perfect ([3]; or [1, Theorem 2.1]), and prime because, using Cohen–Macaulayness, it is enough to check primeness on the open sets $X_i \neq 0$, where it is obvious (this argument actually proves that the ideal of minors $I_{R_{g-1}}$ is the ideal of forms vanishing on R_{g-1}).

From this it follows that $F[X_0, \ldots, X_{g-2}]/(I_{R_{g-1}}, Q)$ is a Cohen–Macaulay ring, and thus that $(I_{R_{g-1}}, Q)$ is unmixed, as required, so $(I_{R_{g-1}}, Q) = I_{C_1}$. Since the degree of $S \cap V$ is $2g+2$, we see further that $S \cap V \cap W$ consists of exactly $2g+2$ points, and thus is the set of Weierstrass points of C_1.

To prove the converse, note that for V to meet S transversely, it must miss the vertex of S, so projection from the vertex will be a degree 2 map $\sigma : V \cap S \to R_{g+1} \cong \mathbb{P}^1$; thus $V \cap S$ is hyperelliptic. Since a hyperplane through the vertex of S meets S in $g+1$ lines, and each of these lines meets $V \cap S$ in a fiber of σ, we see that $V \cap S$ is hypercanonically embedded. □

The moduli of hyperelliptic curves are given by the images in $\mathbb{P}^1$ of the $2g+2$ Weierstrass points under $\sigma : C \to \mathbb{P}^1$ (the Weierstrass points are the ramification points of σ). These images may be taken to be 0, 1, ∞, and $2g-1$ other distinct points $(s : t) \in \mathbb{P}^1$, uniquely determined up to an action of the permutation group on $2g+2$ letters (we chose which points to call 0, 1, ∞.) The next result shows how one can read this information from the hypercanonical embedding:

Corollary 2. *Let*

$$\left(\det\begin{pmatrix} X_0 & X_1 & \cdots & X_g \\ X_1 & X_2 & \cdots & X_{g+1} \end{pmatrix}, Q\right)$$

be the ideal of the hypercanonical curve C_1 as in Theorem 1. The points $(s : t) \in \mathbb{P}^1$ over which σ ramifies are those for which

$$Q(s^{g+1}, s^g t, \ldots, st^g, t^{g+1}, 0) = 0.$$

In suitable coordinates we may write Q in the form

$$Q = X_{g+2}^2 + X_0 p(X_1 \cdots X_g) + X_{g+1} q(X_0 \cdots X_g),$$

where p and q are linear forms in the indicated variables satisfying the linear relations $l(1,0,\ldots,0) = 1$ and $l(1,\ldots,1) + k(1,\ldots,1) = 0$. Thus C_1 depends on $2g-1$ parameters.

Proof. This follows at once from the construction of Q in the proof of Theorem 1 (the linear relations on l and k come from the conditions that 0 and 1 be among the points over which σ ramifies). It is perhaps amusing to note that Q could easily be reduced to the given form even if we did not know its origin in the equation $u^2 = \prod (t - t_i)$. For modulo the equations of $I_{R_{g-1}}$, we could eliminate all but terms of the form $X_0 X_i$ and $X_i X_{g+1}$ $(i = 0, \ldots, g+1)$ from any quadratic form in $X_0, \ldots, X_{g+1}$. □

2. Scrolls and the transcanonical embeddings

I am grateful to Joe Harris for his help with the material of this section; he not only taught me what scrolls were, but also showed me how to finish the proof of the Theorem below.

We recall the facts about rational normal scrolls of dimension 2, which for brevity we simply call *scrolls:*

Fix two integers $a_1, a_2 \geq 0$, at least one $\neq 0$. The tautological line bundle $\mathcal{O}_E(1)$ on the projectivised vector bundle

$$E = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(-a_2)) \to \mathbb{P}^1$$

defines a map $E \to \mathbb{P}^{a_1+a_2+1}$ whose image is by definition the scroll $S(a_1, a_2)$ (see [5, V 2.19]). $S(a_1, a_2)$ is thus a nondegenerate surface, nonsingular if $a_1, a_2 > 0$ (in this case $S(a_1, a_2) \cong E$). On the other hand $S(a_1, 0)$ is the cone over the rational normal curve R_{a_1}.

Alternatively, $S(a_1, a_2)$ may be described by choosing, in $\mathbb{P}^{a_1+a_2+1}$, disjoint linear subspaces W_1, W_2 of dimensions a_1 and a_2 respectively, and letting $\lambda_i : \mathbb{P}^1 \to W_i$ be the a_i-uple embedding, with image the rational normal curve of degree a_i. Then $S(a_1, a_2)$ is the surface swept out by the lines $L_t = \overline{\lambda_1(t)\lambda_2(t)}$ as t varies over $\mathbb{P}^1$. From this description, it is clear that the degree of $S(a_1, a_2)$ is $a_1 + a_2$.

The ideal of forms vanishing on $S(a_1, a_2)$ may in suitable coordinates $X_0, \dots, X_{a_1}$, $Y_0, \dots, Y_{a_2}$ be written as $I_{a_1,a_2} = \det \varphi$, where φ is the $2 \times (a_1 + a_2)$ matrix

$$\varphi_{a_1a_2} = \begin{pmatrix} X_0 & X_1 & \cdots & X_{a_1-1} & Y_0 & Y_1 & \cdots & Y_{a_2-1} \\ X_1 & X_2 & \cdots & X_{a_1} & Y_1 & Y_2 & \cdots & Y_{a_2} \end{pmatrix}.$$

To check this, note first that $F[X_0, \dots, X_{a_1}, Y_0, \dots, Y_{a_2}]/\det \varphi_{a_1a_2}$ is Cohen–Macaulay by the generic perfection of determinantal ideals, and thus $\det \varphi_{a_1a_2}$ is unmixed; it then suffices to check that $\det \varphi_{a_1a_2}$ coincides with the ideal of functions vanishing on $S(a_1, a_2)$ after restriction to an open set $X_i \neq 0$ or $Y_i \neq 0$, where everything becomes obvious from the second description of $S(a_1, a_2)$, above.

We will also need to know the divisor class group of $S(a_1, a_2)$, which is easily computed from the first description, above, as in [5, V 2.3]: It is free abelian on the two generators H, the hyperplane section, and L the class of a line L_t, the "ruling." The intersection form is easily seen to be $H^2 = a_1 + a_2$, $H \cdot L = 1$, $L^2 = 0$.

We will also make use of the sheaf associated to L: it is the sheaf associated to the graded module $\mathcal{F}$ which is the cokernel of the map of free graded $F[X_0 \cdots Y_{a_2}]$-modules whose matrix is the transpose of φ_{a_1,a_2}; that is, $\mathcal{F} = \operatorname{coker} \varphi^*_{a_1,a_2}$.

We write $\operatorname{Symm}_k \mathcal{F}$ for the kth symmetric power of $\mathcal{F}$; the associated divisor is of course kL.

Fixing these notations, we return to the hyperelliptic curve C and describe the kth transcanonical image $C_k \subset \mathbb{P}^{g+2k}$, for $k \geq 1$.

Theorem 3. *The curve $C_k \subset \mathbb{P}^{g+2k}$ is a divisor of type $2H - (2k-2)L$ on the scroll*

$S(g+k,k-1)$. The ideal of forms vanishing on C_k is the ideal $I_{g+k,k-1}$ of the scroll together with the image of a monomorphism of $(\mathrm{Symm}_{2k-2}\,\mathscr{F})(-2)$ *into $F[X_0,\ldots,X_{g+k},Y_0,\ldots,Y_{k-1}]/I_{g+k,k-1}$; in particular, it is minimally generated by the $\binom{g+2k-1}{2}$ quadratic forms generating $I_{g+k,k-1}$ together with $2k-1$ more quadratic forms.*

Remarks. (1) The first statement above could be paraphrased by saying that there is a quadric hypersurface meeting $S(g+k,k-1)$ in the union of C and $2k-2$ lines.

(2) It is possible to give an explicit form for the $2k-1$ quadratic forms that generate the image of $\mathrm{Symm}_{2k-2}\,\mathscr{F}$, and even to describe all the syzygies of the ideal of forms vanishing on C_k, using the techniques of [1, 2]. In fact, the same techniques yield such results for all the arithmetically Cohen–Macaulay divisors on rational normal scrolls (of all dimensions); we hope to devote a future paper to these topics.

Proof of Theorem 3. Writing the affine equations for C as $u^2=\prod_1^{2g+2}(t-t_i)$ as in Theorem 1, and taking K_0 to be the sum of the two points at infinity one sees easily that the functions $1,t,\ldots,t^{g+k},u,ut,\ldots,ut^{k-1}$ form a basis for the space of rational functions whose polar divisors are $\le(g+k)K_0$. Since the minors of the matrix

$$\begin{pmatrix} 1 & t & \cdots & t^{g+k-1} & u & ut & \cdots & ut^{k-2} \\ t & t^2 & \cdots & t^{g+k} & ut & ut^2 & \cdots & ut^{k-1} \end{pmatrix}$$

all vanish identically, it follows that C_k is a divisor on the scroll $S=S(g+k,k-1)$. The points on C for which t takes on a given value are clearly the points lying on a given ruling L_t of the scroll, and for $t\ne t_i$ $(i=1,\ldots,2g+2)$ there are 2 of these. Thus $C\cdot L=2$, and from the form of the intersection pairing on S we see that $C_k\sim 2H-m$ for some m. A computation of the degree $(2H-mL)\cdot H$ shows $m=2k-2$.

Writing $\tilde{\ }$ for the functor associating a sheaf on $\mathbb{P}^{g+2k}$ to a graded $F[X_0,\ldots,Y_{k-1}]$-module, we see that C_k is the locus of zeros of a section of $\mathrm{Symm}_{2k-2}\,\tilde{\mathscr{F}}^{-1}(2)$, so the sheaf of ideals $\mathscr{I}_{C_k}$ defining C_k in S is the image of a monomorphism

$$(\mathrm{Symm}_{2k-2}\,\tilde{\mathscr{F}})(-2)\to\mathcal{O}_S.$$

Since S is arithmetically Cohen–Macaulay, it follows that the ideal of forms in the homogeneous coordinate of S that vanish on C_k is $\sum_{n\ge 0}H^0(\mathscr{I}_{C_k}(n))$, which is isomorphic to $\sum_{n\ge0}H^0((\mathrm{Symm}_{2k-2}\,\tilde{\mathscr{F}})(-2)(n))$. But $\mathrm{Symm}_{2k-2}\,\mathscr{F}$ is a Cohen–Macaulay module of maximal dimension over the homogeneous coordinate ring of S [1, p. 204], so the induced map

$$(\mathrm{Symm}_{2k-2}\,\mathscr{F})(-2)\to F[X_0\cdots X_{g+k},Y_0\cdots Y_{k-1}]/I_{g+k,k-1}$$

is a monomorphism. The result on the number of quadrics generating I_{C_k} follows since $\mathscr{F}$ is minimally generated by 2 elements (of degree 0), and thus $\mathrm{Symm}_{2k-2}\,\mathscr{F}(-2)$ is minimally generated by $2k-1$ elements of degree 2. □

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