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A simpler proof of the Gieseker-Petri Theorem on special divisors

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Introduction

Throughout this paper we work over the complex numbers.

Let C be a smooth projective curve with canonical bundle K, and let L be a line bundle on C. We say that L *satisfies Petri's Condition* if the natural map

$$\mu\colon H^0(L)\otimes H^0(KL^{-1})\to H^0(K)$$

is injective. We say that C satisfies Petri's Condition if this is true for every line bundle L on C.

In this paper we give a new proof of the Gieseker-Petri Theorem:

Theorem A. *Almost every curve satisfies Petri's Condition.*

In a family of curves, Petri's condition is open, so it is enough to exhibit one smooth curve of each genus satisfying the condition.

The Theorem is central in the theory of divisors on a general curve C of genus g. Some of its consequences, established in [1] and [4] (see also [2]) are:

(*) The variety $G^r_d(C)$ of linear systems of degree d and dimension r is smooth and irreducible of dimension $\rho := g-(r+1)(g-d+r)$; it is a "rational resolution" of singularities of the variety $W^r_d(C)$ of line bundles L of degree d on C satisfying $h^0(L)\geqq r+1$, and $W^r_d(C)$ is singular precisely along W^{r+1}_d.

(*) If $\phi\colon C\to \mathbb{P}^r$ is any nondegenerate map and $N_\phi=\phi*\theta_{\mathbb{P}^r}/\theta_C$ is the normal bundle, then $H^1(N_\phi)=0$.

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(*) More trivially, if L is any line bundle on C with $h^0(L) \geqq 2$ then $H^1(L^2) = 0$.

To see the plausibility of the Theorem, we may think of μ locally as a $g \times (r+1)(g-d+r) = g \times (g-\rho)$ matrix of analytic functions of C and $L \in W_d^r(C)$ and note that the set of pairs (C, L) for which μ is not injective is defined by the $(g-\rho) \times (g-\rho)$ minors of this matrix. Such a determinantal variety "should" have codimension $\rho+1$, so that in the simplest of worlds, it would fail to meet $W_d^r(C)$ for a generic curve C, and the set of curves failing to satisfy Petri's Condition would form a divisor in the moduli space. It is not hard to see that this really does happen if g is reasonably small; this is presumably the basis for Petri's offhand remark in [7] that Theorem A "is known to hold".

Theorem A was first proved by Gieseker in [4], using a rather complex argument centering on the degeneration of a smooth curve to a nodal rational curve. Gieseker's degeneration is a sort of stable version of the degeneration used by Griffiths and Harris [6] to prove the "Brill-Noether Formula" $\dim W_d^r(C) = \rho$ for a general curve C of genus g. In [3] we found a far simpler proof of the Brill-Noether Formula by specializing to cuspidal, rather than nodal, rational curves, so we naturally hoped that similar techniques would simplify the proof of Theorem A. (The price, here as in [3], is the loss of validity in positive characteristic, as compared with [6] and [5].) This paper is the issue. Given the degeneration chosen here, much of the technical complication of Gieseker's, argument becomes unnecessary. New ideas centering on the ramification of the limit of the canonical series are required, but the plan of attack in our proof is based on that of Gieseker.

Since the degenerate curve adopted in this paper is not a cuspidal rational curve, we say a word about its genesis from [3]: First, to avoid problems with base change we replaced the g-cuspidal rational curve of Fig. 1a with its "stable" form, a rational curve with g elliptic curves attached to it as in Fig. 1b.

Since, unlike the Brill-Noether Formula, the Petri Condition fails for some curves of this type (the example in [3, Sect. 9] is valid here as well), a further degeneration, as in [3] or [6] is suggested. Accordingly, we let the points of attachment come together one by one. The corresponding stable reduction of this family, after perhaps further base changes and blowups of the resulting singular points, is a curve of the form X_0 shown in Fig. 1c, the notation of which we shall use throughout this paper: X_0 consists of a chain of rational curves $Y_1, \ldots, Y_N$, such that Y_{i-1} and Y_i meet normally at a point p_i; g of the Y_i,

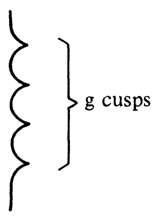

Fig. 1a

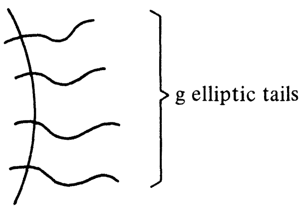

Fig. 1b

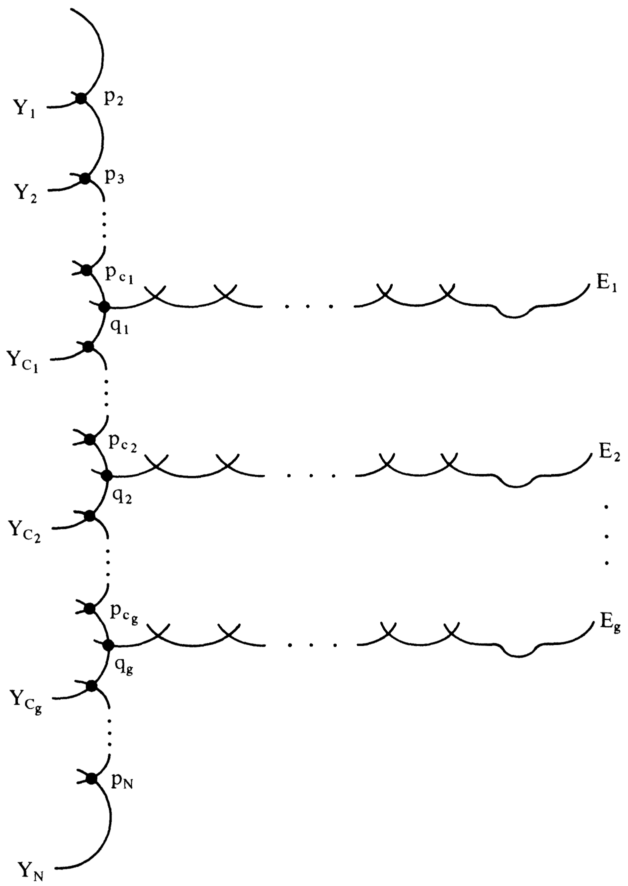

Fig. 1c. Components are smooth and meet transversely as shown. The E_i are elliptic. All other components are rational

say $Y_{c_1}, \ldots, Y_{c_g}$, carry chosen points $q_i \in Y_{c_i}$ at which are attached further chains of rational curves, ending in elliptic curves E_i.

We shall prove that Petri's condition holds for all smooth curves (necessarily of genus g) "near" X_0:

Theorem A'. *If X is a smooth variety and π: $X \to T$ is a flat projective family with fiber X_0 over a point $0 \in T$ as in* Fig. 1c, *then for all t in a neighborhood of* 0 *the fiber X_t satisfies Petri's condition.*

Since families satisfying the conditions of the theorem exist [8] (and can be constructed easily as in [3], appendix to Sect. 5), this will suffice to prove Theorem A.

We now outline the proof of Theorem A':

First, we can and will assume that T is the spectrum of a discrete valuation ring $\mathcal{O}$ with closed point 0 and generic point η, and we must prove Petri's condition for the geometric generic fiber $X_{\bar{\eta}} = X \otimes_{\mathcal{O}} \overline{k(\eta)}$.

Any line bundle on $X_{\bar{\eta}}$ comes from a line bundle defined over some finite extension k' of $k(\eta)$. But if we make a finite base change $T' \to T$ and minimally resolve the singularities of the pulled-back family X', we find that the fiber X'_0 over $0 \in T'$ has the same general form as X_0. Thus, after a change of notation, it suffices to prove Petri's Condition for every line bundle $\mathscr{L}_\eta$ on $X_\eta = X \otimes_{\mathscr{O}} k(\eta)$.

To connect this problem with the geometry of X_0, we shall take the "limit" of $\mathscr{L}_\eta$. For our purposes at least, *this limit consists of a linear series V_Y on each component Y of X_0*. It arises as follows:

The Jacobian of X_0 is simply the product of the Jacobians of the components of X_0. In particular, it is compact, so $\mathscr{L}_\eta$ can be extended to a line bundle on all of X. Since the intersection pairing among the components of X_0 is unimodular, we can alter a given extension of $\mathscr{L}_\eta$ by tensoring with line bundles associated with components of X_0 to get an extension $\mathscr{L}_Y$ of $\mathscr{L}_\eta$ that has degree 0 on all components of X_0 except Y so that $\deg \mathscr{L}_Y|_Y = \deg \mathscr{L}_Y$. We define $V_Y \subset H^0(\mathscr{L}_Y|_Y)$ to be the image of the restriction map

$$\pi_*(\mathscr{L}_Y) \to \pi_*(\mathscr{L}_Y|_Y) = H^0(\mathscr{L}_Y|_Y);$$

that is, V_Y is the space of sections of $\mathscr{L}_Y$ over Y which can be extended to a neighborhood of X_0. (The ideas of this paper can also be used for limits of incomplete linear series, but we shall not need this.)

The idea of the proof of Theorem A′ to be given in Sect. 3 can now be described. Suppose $\mathscr{L}_\eta$ violates Petri's Condition. If $\mathscr{K}_\eta$ is the canonical bundle of X_η, and if V_Y, W_Y, and K_Y are the linear series on a component $Y \subset X_0$ arising in the limits of L_η, $K_\eta \otimes L_\eta^{-1}$ and K_η respectively, then the maps $V_Y \otimes W_Y \to K_Y$ induced by $\pi_*(\mathscr{L}_Y) \otimes \pi_*(K_Y \otimes \mathscr{L}_Y^{-1}) \to \pi_*(K_Y)$ would all have kernels, and these kernels could be related for different Y. Using such relations for the curves Y_i, and the very precise description possible for the limit of the canonical series one can show that any element of the kernel of $V_{Y_1} \otimes W_{Y_1} \to K_{Y_i}$ would be expressible in terms of elements of V_{Y_i} and W_{Y_1} that vanish to rather high order at the points p_i and p_{i+1} where Y_i meets the rest of X_0; to impossibly high order, as a computation shows, when $i=1$ or N.

Before carrying this out we prove, in Sect. 1, some simple general results about limits of linear series. In Sect. 2. we apply this to give a picture of the limit of the canonical series. Much more could be said in this direction with little effort, but we go only far enough to exhibit the point we need for Theorem A′.

With these preparations, the proof of Theorem A′ given in Sect. 3 is rather easy. It consists of a general construction extending Proposition 1.1 to the limit of a product of two series, and, finally, the computation of vanishing order referred to above.

1. Limits of linear series

We fix a projective map

$$\pi\colon\ X \to T = \operatorname{Spec} \mathscr{O} \ni 0, \eta$$

where $\mathcal{O}$ is a discrete valuation ring with parameter t, special point 0, and generic point η, X is a smooth surface, and $\pi^{-1}(0)$ is a reduced curve with smooth components whose dual graph has no loops (such as the curve of Fig. 1c). We fix a line bundle $\mathcal{L}_\eta$ on the general fiber X_η. Let

$$\{V_Y \subset H^0(\mathcal{L}_Y|_Y) \mid Y \text{ a component of } X_0\}$$

be the limit of $\mathcal{L}_Y$ in the sense above.

Because $\mathcal{L}_Y$ has degree 0 on the components of X_0 other than Y, a section of $\mathcal{L}_Y|_{Y_0}$ vanishing on Y vanishes on all of X_0. Thus we may identify V_Y with $\pi_*(\mathcal{L}_Y) \otimes k(0)$. Since $\pi_* \mathcal{L}_Y$ is a free $\mathcal{O}$-module, we see that $\dim_{k(\eta)} H^0(\mathcal{L}_\eta) = \dim_{k(0)} V_Y$; we call this dimension $r+1$. Of course we also have $\deg(\mathcal{L}_Y|_Y) = \deg(\mathcal{L}_\eta)$; call this degree d.

Let Y and Z be two components of X_0 that meet in a point p, and let F be the connected component of $X_0 - p$ that contains Z. It is easy to see that $\mathcal{L}_Z \cong \mathcal{L}_Y(-dF)$. We will henceforth identify $\mathcal{L}_Z$ with this subsheaf of $\mathcal{L}_Y$; correspondingly, $\pi_* \mathcal{L}_Z$ becomes a free $\mathcal{O}$-submodule of $\pi_* \mathcal{L}_Y$. If $\sigma \in \pi_* \mathcal{L}_Y - t\pi_* \mathcal{L}_Y$ then σ does not vanish as a section of $\mathcal{L}_Y|_Y$, and if $p' \in Y$ is any point we write $\operatorname{ord}_{p'}(\sigma, \mathcal{L}_Y|_Y)$, or simply $\operatorname{ord}_{p'}(\sigma|_Y)$ for its order of vanishing as a section of $\mathcal{L}_Y|_Y$.

We have a fundamental inequality:

Proposition 1.1. *If p' is a point of Y distinct from $p = Y \cap Z$, and α is the unique integer such that*

$$t^\alpha \sigma \in \pi_* \mathcal{L}_Z - t\pi_* \mathcal{L}_Z,$$

then

$$\begin{aligned} \operatorname{ord}_{p'}(\sigma, \mathcal{L}_Y|_Y) &\leqq d - \operatorname{ord}_p(\sigma, \mathcal{L}_Y|_Y) \\ &\leqq \alpha \\ &\leqq \operatorname{ord}_p(t^\alpha \sigma, \mathcal{L}_Z|_Z). \end{aligned}$$

Proof. The first inequality is trivial since

$$d = \deg \mathcal{L}_Y|_Y = \sum_{q \in Y} \operatorname{ord}_q(\sigma, \mathcal{L}_Y|_Y).$$

For the second inequality, note that since $t^\alpha \sigma$ is a section of $\mathcal{L}_Z = \mathcal{L}_Y(-dF)$, it vanishes as a section of $\mathcal{L}_Y$ to order $\geqq d$ along F. It follows that σ vanishes as a section of $\mathcal{L}_Y$ to order $\geqq d - \alpha$ along F, and thus $\sigma|_Y$ vanishes to order $\geqq d - \alpha$ at p as a section of $\mathcal{L}_Y|_Y$, as desired.

For the last inequality we use the fact that $\mathcal{L}_Z = \mathcal{L}_Y$ locally along $X_0 - F$, so $t^\alpha \sigma$ vanishes as a section of $\mathcal{L}_Z$ to order $\geqq \alpha$ on $X_0 - F$. Thus $t^\alpha \sigma|_Z$ vanishes to order $\geqq \alpha$ at p as a section of $\mathcal{L}_Z|_Z$. //

If p is a point on a curve Y and V is any linear series on Y with $\dim V = r + 1$, we write

$$a_0(V, p) < \ldots < a_r(V, p)$$

for the $r+1$ distinct orders of vanishing at p of sections in V; we call this the *vanishing sequence of V at p.*

Proposition 1.1 is particularly useful in conjunction with the following "compatible basis" lemma.

Lemma 1.2. *For any point $p' \in Y$ there is a basis $\{\sigma_i\}$ of $\pi_* \mathscr{L}_Y$ such that*

1) $\operatorname{ord}_{p'}(\sigma_i, \mathscr{L}_Y|_Y) = a_i(V_Y, p')$

and

2) *For suitable positive integers α_i the elements $t^{\alpha_i}\sigma_i$ form a basis of $\pi_* \mathscr{L}_Z$.*

Proof. We may apply "Gaussian Elimination" to diagonalize a matrix representing the inclusion $\pi_* \mathscr{L}_Z \subset \pi_* \mathscr{L}_Y$; this gives us a basis satisfying 2). If $\alpha_i \geqq \alpha_j$ and $g \in \mathcal{O}$ then 2) will still hold after replacing σ_i by $\sigma_i + g\sigma_j$, and such transformations, with permutations, suffice for passing to a basis satisfying 1) as well. //

Putting 1.1 and 1.2 together, we get:

Proposition 1.3. *If Y and Z are components of X_0 meeting at p, and p' is a point of Y distinct from p then for $i = 0, \ldots, r$,*

$$a_i(V_Y, p') \leqq a_i(V_Z, p).$$

Further, if equality holds for some i, then there is a section $\sigma \in V_Y$ vanishing to order $a_i(V_Y, p')$ at p' and vanishing on Y only at p' and p.

Proof. Let $\{\sigma_i\}$ be a basis of $\pi_* \mathscr{L}_Y$ satisfying Lemma 1.2. Applying Proposition 1.1 we get $a_i(V_Y, p') \leqq \operatorname{ord}_p(t^{\alpha_i}\sigma, \mathscr{L}_Z|_Z)$, and if equality holds for i then

$$\begin{aligned} \operatorname{ord}_{p'}(\sigma_i, \mathscr{L}_Y|_Y) &= a_i(V_Y, p') \\ &= d\text{-}\operatorname{ord}_p(\sigma_i, \mathscr{L}_Y|_Y), \end{aligned}$$

so that σ vanishes on Y only at p and p'.

There is a permutation f of $\{0, \ldots, r\}$ such that

$$\operatorname{ord}_p(t^{\alpha_i}\sigma_i, \mathscr{L}_Z|_Z) \leqq a_{f(i)}(V_Z, p).$$

The following combinatorial fact completes the argument:

Lemma 1.4. *If $a_0 < \ldots < a_r$ and $b_0 < \ldots < b_r$, and if for some permutation f of $\{0, \ldots, r\}$ we have*

$$a_i \leqq b_{f(i)} \quad \text{for } i = 0, \ldots, r$$

then in fact

$$a_i \leqq b_i \quad \text{for } i = 0, \ldots, r.$$

Further, if for some i we have $a_i = b_i$, then $f(i) = i$ so that $a_i = b_{f(i)}$ as well. //

We now turn to the effect of irrational components of X_0.

We say that a linear series V on a curve Y has at least a cusp at $q \in Y$ if $a_1(V, q) \geqq 2$. Note that this property is inherited by a pencil (2-dimensional subspace) contained in V.

Proposition 1.5. *If $Y, W_1, \ldots, W_l$, is a chain of irreducible components of X_0 as in Fig. 2, with $l \geqq 1$,*

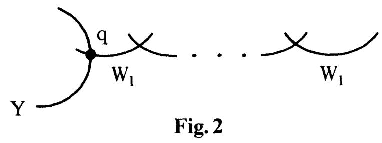

Fig. 2

and W_l is irrational, then V_Y has at least a cusp at $q = Y \cap W_1$.

Proof. We may assume $\dim V \geqq 2$.

If V_Y did not have at least a cusp at q, then it would possess elements vanishing precisely to orders 0 and 1 at q. By 1.3, the vanishing sequence for W_1 at q would end with $d-1, d$. Subtracting $(d-1)p$ from the corresponding pencil in V_{W_1}, we would obtain a g_1^1 on W_1. Of course this implies that $W_1 \cong \mathbb{P}^1$. If W_1 is irrational this is absurd. If $l > 1$, then by induction on l, V_{W_1} has at least a cusp at $q' = W_1 \cap W_2$, which would be inherited by the g_1^1; this is again absurd. //

We will not use 1.4 directly, but rather by means of the following remark: If $p, p', q \in \mathbb{P}^1$ are distinct points, and two linearly independant sections of a line bundle on $\mathbb{P}^1$ are nonvanishing on $\mathbb{P}^1 - \{p, p'\}$, then some linear combination of these vanishes to order exactly 1 at q. (*Proof.* In affine coordinates, with $(p, p') = (0, \infty)$, $x^d - \alpha^d$ has a simple root at $\alpha \neq 0$. // This is our main use of characteristic 0!)

Returning to the notation of Proposition 1.5, and using Proposition 1.3, we have immediately:

Corollary 1.6. *Suppose $Y \cong \mathbb{P}^1$ and that $q = Y \cap W_1$, p, p' are distinct points of Y.*

a) *There is in V_Y, up to scalars, at most one section vanishing only at p and p'.*

b) *If Y meets a component Z of X_0 at p then for all but at most one value of i,*

$$a_i(V_Y, p') < a_i(V_Z, p). \quad //$$

2. The limit of the canonical series

In addition to the hypothesis of section 1 we now assume that X_0 has the form given in Fig. 1c, and we take $L_\eta = K_{X_\eta}$, the canonical sheaf. From the results of section 1 we could exhibit explicit bases for the V_{Y_i}. (Indeed, we could do the same whenever $\rho = g - (r+1)(g-d+r) = 0$). We will actually need only the following result:

Proposition 2.1. *For $m = 1, \ldots, g$, a section in V_{c_m} vanishing on Y_{c_m} only at p_{c_m} and p_{c_m+1} vanishes at p_{c_m} to order exactly $2m-2$.*

Proof. To simplify the notation, set

$$a_i^l = a_i(V_{Y_l}, p_l).$$

By the last statement of Proposition 1.3 it will suffice to show that for each $m = 1, \ldots, g$ there is an i with

$$a_i^{c_m} = a_i^{c_m+1} = 2m-2.$$

But it is quite easy to compute the a_i^l explicitly, as in the next lemma. The following table gives the values for $g=3$ and is typical:

Table 1

i \ l	$l \leqq c_1$	$c_1 < l \leqq c_2$	$c_2 < l \leqq c_3$	$c_3 < l$
0	0	0	1	2
1	1	2	2	3
2	2	3	4	4

To state the result conveniently we make the conventions $c_0=1$, $c_{g+1}=\infty$.

Lemma 2.2. *If $c_{m-1} < l \leqq c_m$ then*

$$a_i^l = \begin{cases} m+i-2 & \text{if } i < m-1 \\ m+i-1 & \text{if } i \geqq m-1 \end{cases}.$$

In particular $a_{m-1}^{c_m} = a_{m-1}^{c_m+1} = 2m-2$.

Proof. Since for each i, l we have

$$0 \leqq a_i^l < a_{i+1}^l \leqq \deg(\mathscr{L}_{Y_l}|_{Y_l}) = 2g-2,$$

it follows that $i \leqq a_i^l \leqq g-1+i$. Thus

$$\sum_i (a_i^N - a_i^2) \leqq g(g-1).$$

On the other hand, by Proposition 1.3 we have $a_i^l \leqq a_i^{l+1}$ for every i, l, while for $l=c_m$, $m=1, \ldots, g$, Corollary 1.6b) implies that the inequality is strict for all but at most one i – that is, in $r=g-1$ cases for each m.

Thus

$$\sum_i (a_i^N - a_i^2) = \sum_{i,l} (a_i^{l+1} - a_i^l) \geqq g(g-1).$$

It follows that equality holds, and we deduce:

1) $a_i^2 = i$ and $a_i^N = g-1+i$ for $i=0, \ldots, g-1$.
2) If $c_{m-1} < l \leqq c_m$ then $a_i^l = a_i^{c_m}$.
3) For each $m=1, \ldots, g$ there is exactly one $i=i(m)$ such that $a_i^{c_m+1} = a_i^{c_m}$.
4) If $i \neq i(m)$ then $a_i^{c_m+1} = a_i^{c_m}+1$.

The assertion of the lemma now amounts to the statement that $i(m)=m-1$ for each m and this follows combinatorially from 1)–4), with the fact that $a_i^l < a_{i+1}^l$ for all i, l. Indeed, $i(g)=g-1$ follows at once from 1) and 3), and dropping the values of a_i^l from consideration if $l \geqq c_g$ or if $i=g-1$, we are done by induction on g. //

3. Proof of the Gieseker-Petri Theorem

We begin with a general result about the limit of the product of two series. We keep the notation and hypotheses of Sect. 1, but now we start with two line bundles, $\mathscr{L}_\eta$ and $\mathscr{M}_\eta$, on X_η. We write v for the valuation on $\mathcal{O}$.

Suppose as in Proposition 1.1 that Y and Z are components of X_0 meeting at p. Consider an element

$$\rho \in (\pi_* \mathscr{L}_Y \otimes \pi_* \mathscr{M}_Y) - t(\pi_* \mathscr{L}_Y \otimes \pi_* \mathscr{M}_Y).$$

As in Sect. 1 we may regard $\mathscr{L}_Z$ and $\mathscr{M}_Z$ as subsheaves of $\mathscr{L}_Y$ and $\mathscr{M}_Y$, and we wish to prove an inequality as in Proposition 1.1.

Since ρ cannot be identified with a section of a bundle on X, we must first define terms like $\operatorname{ord}_p(\rho|_Y)$. (Because of the application in view we must *not* use the natural map

$$\pi_*(\mathscr{L}_Y) \otimes \pi_*(\mathscr{M}_Y) \to \pi_*((\mathscr{L} \otimes \mathscr{M})_Y).)$$

Definition. *If p is any point of a component Y of X_0 then* $\operatorname{ord}_p(\rho|_Y)$ *is defined by the relation:*

$$\operatorname{ord}_p(\rho|_Y) \geqq k$$

iff *ρ is in the linear* span *of t $(\pi_* \mathscr{L}_Y \otimes \pi_* \mathscr{M}_Y)$ and elements of the form $\sigma \otimes \tau$ where $\sigma \in \pi_* \mathscr{L}_Y$, $\tau \in \pi_* \mathscr{M}_Y$ and*

$$\operatorname{ord}_p(\sigma|_Y) + \operatorname{ord}_p(\tau|_Y) \geqq k.$$

Note that $\operatorname{ord}_p(\rho|_Y) \leqq \deg(\mathscr{L}_Y|_Y) + \deg(\mathscr{M}_Y|_Y)$.

Analogously with Proposition 1.1 we have:

Proposition 3.1. *If p' is a point of Y distinct from $p = Y \cap Z$, and α is the unique integer such that*

$$t^\alpha \rho \in (\pi_* \mathscr{L}_Z \otimes \pi_* \mathscr{M}_Z) - t(\pi_* \mathscr{L}_Z \otimes \pi_* \mathscr{M}_Z)$$

then

$$\begin{aligned} \operatorname{ord}_{p'}(\rho|_Y) &\leqq \alpha \\ &\leqq \operatorname{ord}_p(t^\alpha \rho|_Z). \end{aligned}$$

Proof. Let $\{\sigma_i\}$ and $\{\tau_j\}$ be bases of $\pi_* \mathscr{L}_Y$ and $\pi_* \mathscr{M}_Y$ satisfying the conditions of Lemma 1.2, so that, in particular there are integers α_i and β_j such that $\{t^{\alpha_i} \sigma_i\}$ and $\{t^{\beta_j} \tau_j\}$ are bases of $\pi_* \mathscr{L}_Z$ and $\pi_* \mathscr{M}_Z$ respectively.

To calculate $\operatorname{ord}_{p'}(\rho|_Y)$ and α we use:

Lemma 3.2. *If $\rho = \sum f_{ij} \sigma_i \otimes \tau_j$ with $f_{ij} \in \mathcal{O}$, then*

$$\operatorname{ord}_{p'}(\rho|_Y) = \min_{\{i,j \mid v(f_{ij}) = 0\}} \operatorname{ord}_{p'}(\sigma_i|_Y) + \operatorname{ord}_{p'}(\tau_j|_Y),$$

and

$$\alpha = \max_{i,j} \alpha_i + \beta_j - v(f_{ij}).$$

Proof of the Lemma: If $\sigma \in \pi_* \mathscr{L}_Y$ and $\operatorname{ord}_p \sigma \geqq l$ then modulo $t\pi_* \mathscr{L}_Y$ we can write σ as a linear combination of those σ_i with $\operatorname{ord}_{p'}(\sigma_i|_Y) \geqq l$ and similarly for $\tau \in \pi_* \mathscr{M}_Y$. This yields the first equality.

For the second equality note that

$$t^\alpha \rho = \sum (t^{\alpha-\alpha_i-\beta_j} f_{ij})(t^{\alpha_i} \sigma_i \otimes t^{\beta_j} \tau_j).$$

Since $\{t^{\alpha_i} \sigma_i \otimes t^{\beta_j} \tau_j\}$ is a basis of $\pi_* \mathscr{L}_Z \otimes \pi_* \mathscr{M}_Z$, we see that α is the smallest number so that, for every i,j, $t^{\alpha-\alpha_i-\beta_j} f_{ij} \in \mathcal{O}$, whence the desired relation. //

Returning to the proof of 3.1, Proposition 1.1 gives

$$\operatorname{ord}_{p'}(\sigma_i|_Y) + \operatorname{ord}_{p'}(\tau_i|_Y) \leqq \alpha_i + \beta_j,$$

and with Lemma 3.2 this gives the first inequality of Proposition 3.1.

For the second inequality we use the expression

$$t^\alpha \rho = \sum (t^{\alpha-\alpha_i-\beta_j} f_{ij})(t^{\alpha_i} \sigma_i \otimes t^{\beta_j} \tau_j)$$

to conclude that, for some i, j with $v(t^{\alpha-\alpha_i-\beta_j} f_{ij}) = 0$,

$$\operatorname{ord}_p(t^\alpha \rho|_Z) \geqq \operatorname{ord}_p(t^{\alpha_i} \sigma_i|_Z) + \operatorname{ord}_p(t^{\beta_j} \tau_j|_Z),$$

which by Proposition 1.1 is

$$\geqq \alpha_i + \beta_j.$$

Since $v(t^{\alpha-\alpha_i-\beta_j} f_{ij}) = 0$, we have

$$\alpha_i + \beta_j = \alpha + v(f_{ij}) \geqq \alpha,$$

and we are done. //

We are now ready to complete the proof of the Gieseker-Petri Theorem in the form of Theorem A′. To this end suppose that $L_\eta \otimes \mathscr{M}_\eta = \mathscr{K}_\eta$, the canonical sheaf, and that

$$\rho \in \ker(\pi_* \mathscr{L}_\eta \otimes \pi_* \mathscr{M}_\eta \to \pi_* \mathscr{K}_\eta).$$

We must show that $\rho = 0$.

We may identify $\pi_* \mathscr{L}_{Y_1}$ and $\pi_* \mathscr{M}_{Y_1}$ with $\mathcal{O}$-submodules of $\pi_* \mathscr{L}_\eta$ and $\pi_* \mathscr{M}_\eta$ which span these spaces. Further, for each l we identify $\pi_* \mathscr{L}_{Y_{l+1}}$ as a submodule of $\pi_* \mathscr{L}_{Y_l}$, as was done for $\pi_* \mathscr{L}_Z \subset \pi_* \mathscr{L}_Y$, above, and similarly for $\mathscr{M}$.

If $\rho \neq 0$, then there is for each l a unique integer γ_l such that

$$t^{\gamma_l} \rho \in (\pi_* \mathscr{L}_{Y_l} \otimes \pi_* \mathscr{M}_{Y_l}) - t(\pi_* \mathscr{L}_{Y_l} \otimes \pi_* \mathscr{M}_{Y_l}).$$

Of course we have

$$\begin{aligned} \operatorname{ord}_{p_l}(t^{\gamma_l} \rho|_{Y_l}) &\leqq \deg(\mathscr{L}_{Y_l}|_{Y_l}) + \deg(\mathscr{M}_{Y_l}|_{Y_l}) \\ &= \deg(\mathscr{K}_{Y_l}|_{Y_l}) \\ &= 2g-2. \end{aligned}$$

We shall prove that if $l > c_m$ then

$$\operatorname{ord}_{p_l}(t^{\gamma_l} \rho|_{Y_l}) \geqq 2m,$$

which gives a contradiction for $m = g$, showing that $\rho = 0$.

By an obvious induction using Proposition 3.1, it suffices to show that if

$$\operatorname{ord}_{p_{c_m}}(t^{\gamma_{c_m}}\rho|_{Y_{c_m}}) \geqq 2m-2$$

then

$$\alpha = \gamma_{c_m+1} - \gamma_{c_m} \geqq 2m.$$

To simplify the notation, we may suppose $\gamma_{c_m}=0$ so that $t^{\gamma_{c_m}}\rho=\rho$ and we write

$$\begin{aligned} &\alpha \quad \text{for } \gamma_{c_m+1}-\gamma_{c_m}=\gamma_{c_m+1}\\ &Y \quad \text{for } Y_{c_m}\\ &Z \quad \text{for } Y_{c_m+1}\\ &p' \quad \text{for } p_{c_m}\\ &p \quad \text{for } p_{c_m+1}. \end{aligned}$$

We further choose bases $\{\sigma_i\}$ and $\{\tau_i\}$ of $\pi_*\mathscr{L}_Y$ and $\pi_*\mathscr{M}_Y$ satisfying Lemma 1.2, so that in particular there are α_i and β_j so that $\{t^{\alpha_i}\sigma_i\}$ and $\{t^{\beta_j}\tau_j\}$ are bases of $\pi_*\mathscr{L}_Z$ and $\pi_*\mathscr{M}_Z$, respectively.

Writing

$$\rho = \sum f_{ij}\,\sigma_i \otimes \tau_j \qquad f_{ij}\in\mathcal{O},$$

and using Lemma 3.2 our hypothesis becomes:

$$\begin{aligned} 2m-2 &\leqq \operatorname{ord}_{p'}(\rho|_Y)\\ &= \min_{\{i,j \mid v(f_{ij})=0\}} \operatorname{ord}_{p'}(\sigma_i|_Y) + \operatorname{ord}_{p'}(\tau_j|_Y), \end{aligned}$$

and we wish to deduce that for some i,j we have

$$\alpha_i + \beta_j - v(f_{ij}) \geqq 2m.$$

In fact we will find such an i,j for which also $v(f_{ij})=0$.

We now use the fact that $\rho\in\ker(\pi_*\mathscr{L}_Y\otimes\pi_*\mathscr{M}_Y \to \pi_*\mathscr{K}_Y)$. Since

$$\textstyle\sum_{\{i,j \mid v(f_{ij})=0\}} (f_{ij}\,\sigma_i\,\tau_j)|_Y = 0$$

in $H^0(Y,\mathscr{K}_Y|_Y)$ we see that $\operatorname{ord}_{p'}(\rho|_Y)$, which is the minimum order of vanishing at p' of the terms in the sum by Lemma 3.2, must be the order of vanishing of at least two such terms; that is, there are distinct pairs (i_1,j_1) and $(i_2 j_2)$ such that if $(i,j)=(i_1,j_1)$ or (i_2,j_2) then

$$v(f_{ij}) = 0$$

and

$$\operatorname{ord}_{p'}(\rho|_Y) = \operatorname{ord}_{p'}(\sigma_i|_Y) + \operatorname{ord}_{p'}(\tau_j|_Y).$$

Since the orders of vanishing of the σ_i at p' are distinct, and similarly for the τ_j, we must have $i_1 \neq i_2$ and $j_1 \neq j_2$.

It will now suffice to show that for at least one of these two pairs i,j, we have $\alpha_i + \beta_j \geqq 2m$.

By Proposition 1.1, for each i

$$\operatorname{ord}_{p'} \sigma_i \leqq \alpha_i,$$

and if equality holds then σ_i vanishes on Y only at p' and p, so that by Corollary 1.6, equality can hold for at most one i. Similar remarks hold for the β_j, so we obviously have

$$\alpha_i + \beta_j \geqq \operatorname{ord}_{p'} \sigma_i + \operatorname{ord}_{p'} \tau_j + 2 \geqq 2m$$

for $(i,j)=(i_i,j_1)$ or (i_2,j_2), as required, except possibly in the case where (after reversing 1 and 2 if necessary)

$$\operatorname{ord}_{p'} \sigma_{i_1} = \alpha_{i_1} \quad \text{and} \quad \operatorname{ord}_{p'} \tau_{j_2} = \beta_{j_2}.$$

Note that even in this case, $\alpha_{i_1} + \beta_{j_1} \geqq \operatorname{ord}_{p'} \sigma_{i_1} + \operatorname{ord}_{p'} \tau_{j_1} + 1$. Further, since σ_{i_1} and τ_{j_2} vanish on $Y = Y_{c_m}$ only at p and p' we get by Proposition 2.1,

$$\begin{aligned} &\operatorname{ord}_{p'} \sigma_{i_1} + \operatorname{ord}_{p'} \tau_{j_2} \\ &= \operatorname{ord}_{p'} \sigma_{i_1} \tau_{j_2} \\ &= 2m - 2. \end{aligned}$$

But now

$$\begin{aligned} 2m - 2 &\leqq \operatorname{ord}_{p'}(\rho|_Y) \\ &= \operatorname{ord}_{p'} \sigma_{i_1} + \operatorname{ord}_{p'} \tau_{j_1} \\ &\quad + \operatorname{ord}_{p'} \sigma_{i_1} + \operatorname{ord}_{p'} \tau_{j_2} \\ &= 2m - 2, \end{aligned}$$

so $\operatorname{ord}_{p'} \sigma_{i_1} + \operatorname{ord}_{p'} \tau_{j_1} \geqq 2m - 1$, whence $\alpha_{i_1} + \beta_{j_1} \geqq 2m$ as required. //

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