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When Ramification points meet

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Summary. This paper contains three applications of the technique of limit series (our [1986]) to the theory of ramification of linear series on smooth curves, and curves of compact type, over $\mathbb{C}$.

Let $\{L_t \mid |t|<\varepsilon\}$, be a family of linear series on a smooth family of smooth curves $\{C_t\}$, and let $p_1(t), p_2(t) \in C_t$ be sections of the family which coincide (only) at $t=0$. Set $p=p_1(0)=p_2(0)\in C_0$.

We first give a condition related to the Schubert calculus which must be satisfied by the ramification series $\alpha^{L_0}(p)$ and the $\alpha^{L_t}(p_i(t))$. We then take up the converse problem: In what ways can a given ramification point arise as a limit? We show that if the ramification point is *dimensionally proper* in the sense of our [1986], then families of every kind allowed by the Schubert calculus condition can actually be constructed. Finally, we prove that dimensional propriety is in a strong sense an open condition, so that ramification points constructed as above are again dimensionally proper.

In the body of the paper we work not with pairs of points, as above, but with arbitrary finite collections of points approaching (possibly) several points of the limit curve. Further, by their nature, the results are valid for families of curves of compact type.

Introduction and results

Let $(C_t, \{p_j(t)\}_{j=1,\ldots,n})$ for $t\in\mathbb{C}$ near 0 be a family of smooth complex curves with n marked points, distinct at least for $t\neq 0$, and let $L_t=(\mathscr{L}_t, V_t)$ be a family of linear series of (projective) dimension r on C_t, so that for each t $\mathscr{L}_t$ is a line bundle on C_t and V_t is a space of global sections of (vectorspace) dimension $r+1$. The *ramification sequence*

$$\alpha^{L_t}(p_j(t)) \qquad 0\leqq \alpha_0^{L_t}(p_j(t))\leqq \ldots \leqq \alpha_r^{L_t}(p_j(t))$$

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is defined to be the unique nondecreasing sequence of integers such that V_t contains a section vanishing to order $\alpha_i^{L_t}(p_j(t))+i$ at $p_j(t)$ for each $i=0, \ldots, r$. Restricting our attention to small t, we may assume that $\alpha^{L_t}(p_j(t))$ is constant for $t \neq 0$.

It is easy to see that for each i we have $\alpha_i^{L_0}(p_j(0)) \geqq \alpha_i^{L_t}(p_j(t))$. Also, if all the p_j have a common limit $p=p_j(0)$, then the *ramification weights*

$$W^{L_t}(p_j(t))=\sum_{i=0}^{r} \alpha_i^{L_t}(p_j(t))$$

are easily shown to satisfy

$$W^{L_0}(p) \geqq \sum_j W^{L_t}(p_j(t)).$$

But in fact there are more subtle restrictions.

To express the result conveniently we introduce some further notation:

For any linear series $L=(\mathscr{L}, V)$ of dimension r on a curve C, and for any $p \in C$, we let $\sigma^L(p)=\sigma_{\alpha^L(p)}$ be the Schubert cycle associated to the index $\alpha^L(p)=(\alpha_0, \ldots, \alpha_r)$ in the cohomology ring of the Grassmannian $G(r+1, \infty)$ of $r+1$-planes. This means that, fixing a flag $\{f^i\}$ of subspaces, with $\operatorname{codim} f^i=i$, in an infinite (or $d+1$) dimensional space, $\sigma^L(p)$ is the cohomology class of the irreducible variety of $r+1$-dimensional subspaces which meet $f^{r+1+\alpha_i-i}$ in a space of dimension $\geqq i$ for $i=0, \ldots, r$. Any cohomology class in $G(r+1, \infty)$ is of course an integral linear combination of the classes $[\sigma_\alpha]$ of Schubert cycles for various indices $\alpha=(\alpha_0 \leqq \ldots \leqq \alpha_r)$, in a unique way. Note that we are writing the indices in nondecreasing order, to match our notation for ramification. (If the reader prefers, he may use the cohomology ring of a sufficiently large finite Grassmannian; if the C_t are complete curves, and the line bundles are of degree d, then $G(r+1, d+1)$ will do.) If $\sigma_{\alpha^1}, \ldots, \sigma_{\alpha^s}$ are Schubert cycles, then the expression

$$\prod_{i=1}^{s} [\sigma_{\alpha^i}]=\sum_\gamma \mathrm{m}_\gamma [\sigma_\gamma]$$

of the product in the cohomology ring has all $m_\gamma \geqq 0$. The Schubert cycles σ_α are partially ordered by inclusion, we have $\sigma_\alpha \subset \sigma_\beta$ iff $\alpha_0 \geqq \beta_0, \ldots, \alpha_r \geqq \beta_r$. *We will extend this order to products* by defining

$$\prod_{i=1}^{s} [\sigma_{\alpha^i}] \supset [\sigma_\beta]$$

if, with notation above, we have $\sigma_\gamma \supset \sigma_\beta$ for some γ with $m_\gamma \neq 0$.

We can now state our first theorem:

Theorem 1. *If, with notation above, $p_1(0)=\ldots=p_s(0)=p$, then for small t*

$$\prod_{j=1}^{s} [\sigma^{L_t}(p_j)] \supset \sigma^{L_0}(p).$$

This generalizes Theorems 7.1 and 8.1 of our [1983], where a proof is given (in the special case where $s=2$ and one of the factors is special) that uses in an essential way the Hodge algebra structure on the coordinate ring of the Grassmannian. The proof to be given below avoids this relatively combinatorial theory by appealing to the geometry of limit series, in the sense of our [1986]. Most of the content of Theorems 7.1 and 8.1 of [1983] which is not explicitly included in the above turn out to follow easily. Thus, combining Theorem 1 with the dimensional transversality results of Sects. 1 and 2 of our [1983] we get for example:

Corollary. *Let $p_1(t), \ldots, p_s(t) \in \mathbb{P}^1$ be points of $\mathbb{P}^1$, distinct for $t \neq 0$, and suppose that for each i, the $p_j(t)$ converge to a point p.*

Fix r and d, and Schubert indices

$$\alpha^j: 0 \leqq \alpha_0^j \leqq \ldots \leqq \alpha_r^j \leqq d-r.$$

If $(p_j(t))$ is the scheme of all $\mathfrak{g}_d^r$'s on $\mathbb{P}^1$ having ramification sequence $\geqq \alpha^j$ at $p_j(t)$, then the limit

$$V: \lim_{t \to 0} \left(\bigcap_j \sigma(p_j(t))\right)$$

is a scheme which, as a cycle, is $\sum m_\gamma \sigma_\gamma$, where

$$\prod_1^s [\sigma_{\alpha^j}] = \sum m_\gamma [\sigma_\gamma]. \quad \square$$

In the special case of Theorem 7.1 and 8.1 of [1983], V is shown to be reduced; it would be nice to prove the analogue here that V is unmixed, or even that V is arithmetically Cohen-Macaulay, but this does not seem to follow from our present methods.

One might well fear that Theorem 1 represents but one of many complex restrictions on the behavior of ramification points in collision; but in fact, under a broad set of circumstances this is not the case. To state the result in its most useful form, we pass to the complete case, and assume that C $(=C_0)$ is a smooth projective curve of genus g, and $L=(\mathscr{L}, V)$ is a $\mathfrak{g}_d^r$ (linear series of dimension r, as above, with $\mathscr{L}$ a line bundle of degree d). Given points $p_1, \ldots, p_r$ on C, we consider the versal deformation $\mathscr{C} \to B$, $\tilde{p}_i: B \to C$ of $(C, p_1, \ldots, p_r)$ and the variety $G_d^r(C/B, \{(\tilde{p}_i, \alpha^L(p_i))\})$ of $\mathfrak{g}_d^r$'s $\tilde{L}$ on C/B such that on each fiber C_b over $b \in B$, the ramification of $\tilde{L}_b$ at $\tilde{p}_i(b)$ satisfies

$$\alpha^{\tilde{L}_b}(\tilde{p}_i(b)) \geqq \alpha^L(p_i)$$

(where the ordering of ramification indices is termwise, as in the relation $\subset$ on Schubert cycles).

We say that $p_1, \ldots, p_n$ are *dimensionally proper ramification points* of L, or that L is *dimensionally proper with respect to* $p_1, \ldots, p_n$ if locally at L the dimension of

$$G_d^r(\mathscr{C}/B, \{(\tilde{p}_i, \alpha^L(p_i))\})$$

is equal to the dimension of the base B plus the "expected dimension" of the fiber, which we call the *adjusted Brill-Noether number*

$$\rho(g, r, d, \{\alpha^L(p_i)\}) =: \rho(L, \{p_i\})$$
$$:= (r+1)(d-g+r) - \sum_1^n w^L(p_i),$$

where the weight $w^L(p_i)$ is as before

$$w^L(p_i) = \sum_{j=0}^r \alpha_j^L(p_i).$$

This notion follows our [1986]. The special case of dimensionally proper Weierstrass points that is the case of the complete canonical series, is studied in our [1987].

The next result is that dimensionally proper ramification points arise as limits in every way allowed by Theorem 1:

Theorem 2. *Let C be a smooth projective curve, and let L be a g_d^r on C, dimensionally proper with respect to points $p_1, \ldots, p_r$. If α^{ij} are Schubert indices,*

$$\alpha^{ij} = 0 \leqq \alpha_0^{ij} \leqq \ldots \leqq \alpha_r^{ij} \leqq d - r$$

such that for $i = 1, \ldots, n$

$$\prod_j [\sigma_{\alpha^{ij}}] \supset [\sigma_{\alpha^L(p_i)}],$$

then there exists a family of curves C_t and linear series L_t with $C_0 = C$, $L_0 = L$, and sections $p_{ij}(t) \in C_t$, with $p_{ij}(0) = p_i$, but $p_{ij}(t)$ distinct, such that $\alpha^{L_t}(p_{ij}(t)) = \alpha^{ij}$ for small t.

It is obvious from the definition that in a family $\mathscr{C}/B$, $\tilde{L}$, p_i: $B \to C$ of g_d^r's on n-pointed smooth curves *such that* $\alpha^{L_b}(p_i(b))$ *is constant*, the set of b such that L_b is dimensionally proper with respect to $p_1(b), \ldots, p_r(b)$ is open. But in fact the restriction given is unnecessary:

Theorem 3. *The dimensionally proper locus is open in any family of g_d^r's on n-pointed smooth projective curves.*

Thus we see that ramification points constructed from dimensionally proper ramification points by the method of Theorem 2 are again dimensionally proper.

It has already been mentioned that the results above extend the material of our [1983], Sects. 7–8. They also play a central role in our study of Weierstrass points in the dimensionally proper domain in [1987]. The proofs depend heavily on the theory developed in our [1986]; in essence, we extend Theorems 1–3 to the case of reducible curves of compact type, where one has semi-stable reduction available to separate ramification points that come together.

Theorems 1–3 are proved in Sect. 2, after we give in Sect. 1 an easy necessary extension of the results of our [1983] to the case of reducible curves of genus 0.

1. Linear series and their ramification on curves of genus 0

Let T be a complete reduced connected curve of arithmetic genus 0. One checks easily that the components of T are smooth and rational, and that the dual graph of T (a node for each component, an edge for each intersection) is a tree. A *crude limit* $\mathfrak{g}_d^r$ (or simply a *crude limit series*) on T is a collection L of $\mathfrak{g}_d^r$'s $L_Y=(\mathscr{L}_Y, V_Y)$ on the components Y of T satisfying the *compatibility conditions:*

whenever components Y and Z meet in a point p,

$$\alpha_i^{L_Y}(p)+\alpha_{r-i}^{L_Z}(p)\geqq d-r.$$

If equality holds in all these conditions, we say that L is a *refined* limit $\mathfrak{g}_d^r$, or simply a limit $\mathfrak{g}_d^r$.

Let $p_1, \ldots, p_s$ be smooths of T. The main result of our [1983, Scts. 1, 2] computes the dimension of the space of linear series on T with prescribed ramification in case T is irreducible. With our present terminology we may extend it to the reducible case, and give it a useful reformulation, as:

Proposition 1.1. *Every limit series L on T is dimensionally proper with respect to any set of smooth points $p_1, \ldots, p_s$. Furthermore, $\prod_1^s [\sigma^L(p_i)] \neq 0$ in the cohomology ring of $G(r+1, d+1)$.*

Conversely given T, $p_1, \ldots, p_s$, and Schubert indices $\alpha^1, \ldots, \alpha^s$ such that

$$\prod [\sigma_{\alpha^i}] \neq 0$$

in the cohomology ring of $G(r+1, d+1)$, there are limit series L on T with $\alpha^L(p_i) = \alpha^i$ for $i=1, \ldots, s$.

Proof. The irreducible case is equivalent to Theorem 2.3 of our [1983]; the general case follows easily by induction on the number of components, using the additivity of the adjusted Brill-Noether number as in Sect. 3 of our [1986]. □

Remarks. 1) It follows easily from the same ideas that the limit series on T form an open, dense subset of the crude limit series.

2) The result can easily be extended to the case where T is of compact type and has components of genus $\leqq 2$ provided that at most one node of T lies on any component of genus >0, no node of T is at a Weierstrass point of a component of genus >0, and all the p_i lie on rational components or at points of elliptic components which are not rationally related to the nodes; see our [1983], Sect. 1.

2. Proof of Theorems 1–3

Proof of Theorem 1. We may begin by repeatedly blowing up our family at limit points of the $p_i(t)$ as $t \to 0$, and making base changes, as in our [1986], Sect. 1 (or as in the process of semistable reduction for pointed curves, see Fig. 2.1) to reach a situation where:

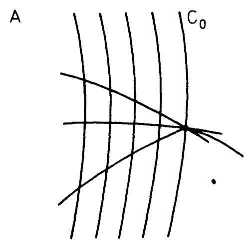
1) C_0 has been replaced by a curve of the form $C_0' = C_0 \cup T$, where T is a complete, reduced, connected curve of genus 0, p is a smooth point of T, and C_0 and T meet transversely at p. In addition to L_0 on C_0 we have a limit $g_d^r L^T$ on T, with $d = \alpha_r^{L_0}(p) + r$ and $\alpha_i^{L^T}(p) = d - \alpha_{r-i}^{L_0}(p)$ for $i = 0, \dots, r$. (If the C_t are complete, this will be the T aspect of the usual limit series on C_0' with base points at p of the aspects removed.)

2) The limits of the $p_i(t)$ as $t \to 0$ are distinct smooth points $p_i'(0)$ of T, and if L_0' is the limit series $\{L_0, L^T\}$ on C_0', the ramification series of L_0' at $p_i'(0)$ satisfy

$$\alpha^{L_0'}(p_i'(0)) = \alpha^{L_t}(p_i(t))$$

for small t.

The theorem now follows from Proposition 1.1 applied to $p, p_1, \dots, p_s \in T$ and the next lemma, taking $\alpha = \alpha^{L_0}(p)$ and $\alpha^i = \alpha^{L^T}(p_i)$.



Blow up p and base-change repeatedly to get:

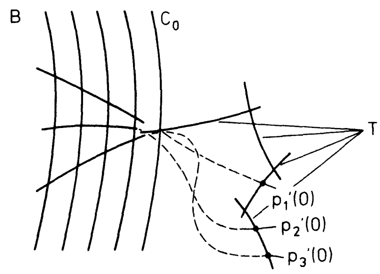


Fig. 2.1

Lemma 2.1. *Let*

$$\alpha\colon 0 \leqq \alpha_0 \leqq \dots \leqq \alpha_r \leqq d-r$$
$$\alpha^j\colon 0 \leqq \alpha_0^j \leqq \dots \leqq \alpha_r^j \leqq d-r, \quad j = 1, \dots, s$$

be Schubert indices, and let

$$\beta\colon 0 \leqq \beta_0 \leqq \dots \leqq \beta_r \leqq d-r$$

be defined by $\beta_i = d - r - \alpha_{r-i}$ $(i = 0, \dots, r)$.

We have

$$\prod_1^s [\sigma_{\alpha^i}] \supset [\sigma_\alpha]$$

in the cohomology ring of $G(r+1,\infty)$ (*or* $G(r+1,d+1)$) *if and only if*

$$[\sigma_\beta]\cdot\prod_1^s [\sigma_{\alpha^i}] \neq 0$$

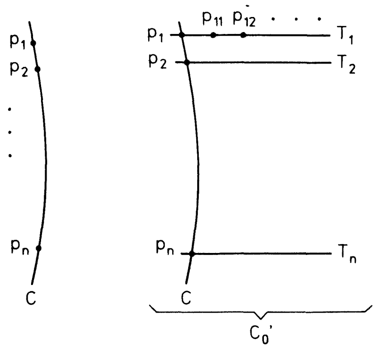
in the cohomology ring of $G(r+1,d+1)$.

Proof of Lemma 2.1. The condition

$$\prod_1^s [\sigma_{\alpha^i}] \supset \sigma_\alpha$$

says that for some σ_γ occurring nontrivially in the standard expression of the left-hand product, $\gamma \leqq \alpha$ termwise. Since the α_i are all $\leqq d-r$, this is true in $G(r+1,\infty)$ if and only if it is true in $G(r+1,d+1)$. But in $G(r+1,d+1)$, σ_β is dual to σ_α, and the lemma follows (see for example the treatment of the Schubert calculus in Griffiths-Harris [1978] pp. 197-207 for background). □

Proof of Theorem 2. We begin by replacing C by the union C_0' of C and n smooth rational curves T_i, meeting C transversely at the points p_i, each carrying marked points p_{ij} distinct from p_i:



For $i=1,\dots,n$, let β^i be the Schubert index defined by

$$\beta^i_k = d-r-\alpha^L_{r-k}(p_i) \qquad k=0,\dots,r.$$

By Proposition 1.1 and Lemma 2.1 there exists a dimensionally proper $\mathfrak{g}^r_d$ L^{T_i} on T_i with

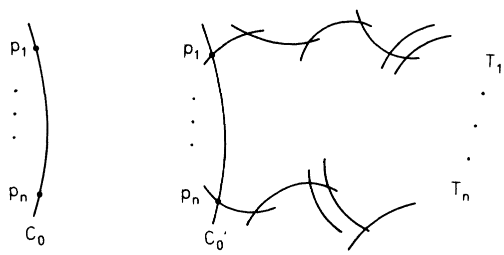
$$\alpha^{L^{T_i}}(p)=\beta^i$$
$$\alpha^{L^{T_i}}(p_{ij})=\alpha^{ij}$$

for all i,j. Together with L on C, the L^{T_i} form the aspects of a limit series on C_0', dimensionally proper with respect to the points p_{ij}. Thus by the smoothing result, Corollary 3.7 of our [1986], there exists a family of curves C_t as desired in Theorem 2, but with C'_0 as central fiber in place of C. We may apply semistable reduction (in the sense of non-pointed curves!) to this family to obtain the desired family with central fiber C. □

Proof of Theorem 3. Since the dimensionally proper locus is obviously constructible, it is enough to show that if $p_1(t), \ldots, p_n(t) \in C_t$ is a 1-parameter family of curves, for small t, and if L_t is a family of g^r_d's on the C_t; such that L_t is dimensionally proper with respect to $p_1(t), \ldots, p_n(t)$ for $t=0$, then the same is true for t near 0. As was remarked in the introduction, this is clear if $\alpha^{L_t}(p_i(t))$ is constant and the $p_i(t)$ are distinct for all t, and we can even assume that these conditions hold for $t \neq 0$.

By the Plücker formulas (eg. Proposition 1.1 of our [1983]) the total weight of all the ramification points of L_t on C_t is constant, so by semicontinuity of the weight, $\alpha^{L_0}(p_i(0)) \neq \alpha^{L_t}(p_i(t))$ (for small t) iff ramification point of L_t other than $p_i(t)$ approaches $p_i(0)$. Thus after sufficiently many blowups and base-changes, in the style of Sect. 2 of our [1986], we may arrive at a family where

1) C_0 is replaced by a union C'_0 of C_0 and trees T_i of rational curves meeting C_0 transversely in p_i



2) Distinct ramification points of L_t on C_t remain distinct on C'_0, and C'_0 with its ramification points is a stable curve; thus ([1986, Proposition 2.5]).

3) The limit of the L_t on C'_0 is a limit series L'_0 on C'_0, and thus, writing $p'_i(0)$ for the limit of $p_i(t)$ on C'_0,

4) $\alpha^{L_t}(p_i(t)) = \alpha^{L'_0}(p'_i(0))$.

Of course the C_0-aspect of L'_0 is just L_0, so by Proposition 1.1, and the hypothesis that L_0 is dimensionally proper with respect to the $p_i(0)$, L'_0 is dimensionally proper with respect to the $p'_i(0)$. It follows as in [1986] Sect. 3, that if $\mathscr{C}/B$ is a versal family of curves around C'_0, as a curve with all its ramification points marked, then with obvious notation $G^r_d(\mathscr{C}/B, \{(p_i, \alpha^{L'_0}(p'_i(0)))\})$ has dimension equal to $\dim B$ plus the adjusted Brill-Noether number of C'_0 with respect to the $p'_i(0)$. Thus in an open neighborhood of (C_0, L_0) all the (C_b, L_b) will be dimensionally proper with respect to $p_i(b)$, and this includes the original family. □

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