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Journal of Pure and Applied Algebra 201 (2005) 284 – 294
JOURNAL OF PURE AND APPLIED ALGEBRA
www.elsevier.com/locate/jpaa

A finiteness property of infinite resolutions☆

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Received 3 January 2003; received in revised form 27 May 2004
Available online 2 March 2005

Dedicated to Prof. Wolmer Vasconcelos on the occasion of his 65th birthday

Abstract

In this paper we prove a finiteness result for infinite minimal free resolutions over a Noetherian local ring R: If M is a module, such as the residue field, that is locally free of constant rank on the punctured spectrum of R, and $I \subset R$ is an ideal, then the maps $f_n : F_n \to F_{n-1}$ in the minimal free resolution of M satisfy the *uniform Artin–Rees property*: $I^N F_{n-1} \cap \operatorname{Im} f_n \subset I^{N-q} \operatorname{Im} f_n$ with Artin–Rees exponent q independent of n. We ask whether the same result holds for any finitely generated module, and we study some related finiteness questions.

MSC: 13C; 13D02; 13D07; 13H

0. Introduction

Let $(R, \mathbf{m})$ be a Noetherian local ring (or a standard graded ring over a field, with $\mathbf{m} = R_+$), and let M be a finitely generated (graded) R-module. Such a module can be defined using finitely many elements of R, for example by giving a free presentation. In the graded case, the entire definition of R and M involves only finitely many coefficients. This paper grew out of an attempt to understand what sorts of finiteness properties might be inherited by

☆ Partially supported by the NSF. The second author was partially supported by DMS-0244405

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0022-4049/$ - see front matter © 2005 Published by Elsevier B.V.
doi:10.1016/j.jpaa.2004.12.048

the—in general infinite—minimal free resolution

$$\mathbf{F} : \cdots \longrightarrow F_n \xrightarrow{f_n} F_{n-1} \xrightarrow{f_{n-1}} \cdots \longrightarrow F_0$$

of M. As a first example, consider the question asked in Eisenbud–Totaro–Reeves [7]:

Question A. Suppose R is standard graded. Are there bases for the F_n such that the matrices expressing the f_n with respect to these bases have entries of bounded degree?

We do not know the answer to Question A even for the resolution of the residue class field. Elementary examples given in Section 5 suggest some of the problems that must be overcome to answer it.

One natural way to relax Question A and extend it to the local case is to use the Artin–Rees exponent: if, in the graded setting, a map of free modules $f : F \to G$ can be written as a matrix whose entries have degree at most q, then $\mathbf{m}^N G \cap \operatorname{Im} f \subset \mathbf{m}^{N-q} \operatorname{Im} f$. Since there seems no reason to restrict to $\mathbf{m}$ rather than using an arbitrary ideal I we ask:

Question B. Suppose R is local, and M is a finitely generated R-module with minimal free resolution as above. Is there an integer q such that $I^N F_{n-1} \cap \operatorname{Im} f_n \subset I^{N-q} \operatorname{Im} f_n$ all $N \geqslant q$ and n?

The main theorem of this paper (Theorem 3.1) says that the answer to Question B is positive when M is locally of finite projective dimension of constant rank on the punctured spectrum of R. This condition is satisfied by any module of finite length over any local ring, and by any finitely-generated module with constant rank on the punctured spectrum over a ring with isolated singularity. Using the main result of Huneke [8] it follows from the proof that the same q would work for every ideal I (under mild assumptions on the ring).

When the answer to question B is positive it follows easily that a fixed power of I annihilates $\operatorname{Tor}_j(M, R/I^N)$ for every j and N (one might call this condition “uniform annihilation”). In Proposition 4.1 we show (under mild hypotheses) that the answer to Question B is positive if and only if this vanishing holds along with a weaker Artin–Rees property. Lemma 3.3 shows that, under the hypothesis of Theorem 3.1, a fixed power of the maximal ideal actually annihilates the whole functor $\operatorname{Tor}_i(M, -)$ for all large i, so at least in this case uniform annihilation is the easier part.

One way to use the Artin–Rees exponent is through the approximation property. Let us say that a complex $\mathbf{F}'$ is a perturbation of $\mathbf{F}$ to orders $q_1, q_2, \ldots$ if the free modules of $\mathbf{F}'$ are the same as those of $\mathbf{F}$, while the differential $f'_n : F_n \to F_{n-1}$ is given by $f'_n = f_n + g_n$, where $g_n : F_n \to \mathbf{m}^{q_n} F_{n-1}$. If the q_i are sufficiently large, and $\mathbf{F}$ is a resolution, then the perturbation is also a resolution (see Proposition 1.1). The values of the q_i for which this is true can be bounded in terms of Artin–Rees exponents—see the discussion around Proposition 1.1, below. Thus, a positive answer to Question B (for a given ring R and module M) implies a positive answer to:

Question C. Let $(R, \mathbf{m})$ be a local ring, and let $\mathbf{F}$ above be the minimal free resolution of a finitely generated module M. Is there a number q such that any complex $\mathbf{F}'$ that is a perturbation of $\mathbf{F}$ as above to order $(q, q, \ldots)$ is exact?

Theorem 3.1 implies that the answer to this question is affirmative when M has finite projective dimension of constant rank on the punctured spectrum. In Proposition 2.1 we show that the answer to question C is also affirmative for any module over a Cohen–Macaulay ring. Again, we ask whether it is affirmative in general.

Another approach to finiteness properties of free resolutions is the conjecture of Huneke proved by Eisenbud and Green [6]. This result implies, for example, that not all of the orders of the entries of matrices representing f_n can go to infinity with n. For further results see Wang [12] and Koh–Lee [9]. It would be interesting to know more statements of this type. When the ring R is a complete intersection of low codimension more is known; see for example Eisenbud [5] and Avramov [1,2].

1. Approximating exact sequences

The following result connects Questions B and C.

Proposition 1.1. *Let $(R, \mathbf{m})$ be a Noetherian local ring, and let I be a proper ideal of R. Suppose that*

$$M \xrightarrow{f} N \xrightarrow{g} K$$

is an exact sequence of finitely generated R-modules. Write $G = \mathrm{Im}(g)$ and $F = \mathrm{Im}(f)$. Let r be an integer such that, for all $n \geqslant r$, $I^n K \cap G \subseteq I^{n-r} G$ and $I^n N \cap F \subseteq I^{n-r} F$. Let $\alpha : M \longrightarrow N$ and $\beta : N \longrightarrow K$ be two homomorphisms such that $\alpha(M) \subseteq I^{2r} N$ and $\beta(N) \subseteq I^{2r} K$. If

$$M \xrightarrow{f+\alpha} N \xrightarrow{g+\beta} K$$

is a complex, then it is exact.

The first result of this kind that we know is that of Peskine and Szpiro [10], who proved that a complex that is a sufficiently good $\mathbf{m}$-adic approximation to a resolution has only finite length homology. Exactness was shown by Eisenbud [4], where some other improvements are given as well. Conrad and de Jong ([3, Lemma 3.1]) have proven a sharper version of Proposition 1.1, showing that the bound $2r$ given above can be improved to $r + 1$. The version here admits a very short proof, part of which is based on the original idea of Peskine and Szpiro [10]. We include it for the convenience of the reader.

Proof. Let $u \in \mathrm{Ker}(g+\beta)$. We first claim that $u \in \mathrm{Im}(f+\alpha) + I^r N$. From $(g+\beta)(u) = 0$ it follows that

$$\beta(u) \in I^{2r} K \cap G \subseteq I^r G.$$

Choose $w \in I^r N$ such that $\beta(u) = g(w)$. Then $g(u+w) = 0$ so that $u + w \in \mathrm{Ker}(g) = F$. Write $u + w = f(v)$ for some $v \in M$. Then $u = (f+\alpha)(v) - \alpha(v) - w \in \mathrm{Im}(f+\alpha) + I^{2r} N + I^r N = \mathrm{Im}(f+\alpha) + I^r N$.

The rest of the proof proceeds as in the original lemma of Peskine and Szpiro [10]. We claim that $u \in \operatorname{Im}(f+\alpha)+I^P N$ for all $p \geqslant r$. An application of Krull's intersection theorem then finishes the proof. In fact we prove that if ($u \in \operatorname{Ker}(g+\beta)$ and) $u \in \operatorname{Im}(f+\alpha)+I^P N$ for some $p \geqslant r$, then $u \in \operatorname{Im}(f+\alpha)+I^{p+r}N$. To prove this, we may assume $u \in I^P N$. As above, $\beta(u) \in I^{p+2r}K \cap G \subseteq I^{p+r}G$ so that there exists an element $w \in I^{p+r}N$ with $g(w)=\beta(u)$. Hence $g(u+w)=0$ and so $u+w \in I^p N \cap F \subseteq I^{p-r}F$. Hence there is an element $v \in I^{p-r}M$ with $u+w=f(v)$. Finally, $u=(f+\alpha)(v)-\alpha(v)-w \in \operatorname{Im}(f+\alpha)+I^{p+r}N$. □

2. The Cohen–Macaulay case

We now turn to question C, and answer it affirmatively when R is a Cohen–Macaulay ring. The proof of the next result shows that, in this case, preserving the exactness of an approximating complex is fundamentally a matter of preserving the exactness of the first dim R terms.

Proposition 2.1. *Let $(R, \mathbf{m})$ be a Cohen–Macaulay local ring and let*

$$\mathbf{F} : \cdots \longrightarrow F_n \xrightarrow{f_n} F_{n-1} \xrightarrow{f_{n-1}} \cdots \longrightarrow F_0 \tag{1}$$

be an exact sequence of finitely generated free modules. Then there exists an $N \geqslant 0$ such that if

$$\cdots \longrightarrow F_n \xrightarrow{f_n+\varepsilon_n} F_{n-1} \xrightarrow{f_{n-1}+\varepsilon_{n-1}} \cdots \longrightarrow F_0$$

is a complex and $\varepsilon_i(F_i) \subseteq \mathbf{m}^N F_{i-1}$, then it is exact.

Proof. We use induction on $\dim(R)$. If the dimension is zero, then $\mathbf{m}^k=0$ for some k and if we approximate the maps to this value, clearly nothing changes.

Assume $\dim(R)>0$ and choose a non-zerodivisor c in $\mathbf{m}$. Truncating the complex at F_1, we can think of this truncated sequence $\mathbf{F}_{\geqslant 1}$ as a free resolution of a torsion-free module, namely the image of f_1. In particular c is not a zerodivisor on this module so that tensoring with $R/(c)$ preserves exactness since $\operatorname{Tor}_i^R(\operatorname{Im}(f_1), R/(c))=0$ for $i \geqslant 1$. By induction there is an integer k_0 such that if we approximate this new acyclic sequence by maps with entries in $\mathbf{m}^{k_0}$, the resulting complex is exact. By Proposition 1.1 there is an integer k_1 such that if we approximate the sequence $F_2 \longrightarrow F_1 \longrightarrow F_0$ by maps with entries in $\mathbf{m}^{k_1}$ it will stay exact. Let N be the maximum of k_0 and k_1. We claim this works to approximate the entire sequence so as to preserve exactness. Choose ε_i as in the statement of the proposition. Let $g_i=f_i+\varepsilon_i$. By choice, $\operatorname{Ker}(g_1)=\operatorname{Im}(g_2)$. Applying the induction to the complex

$$\cdots \longrightarrow \overline{F_n} \xrightarrow{g_n} \overline{F_{n-1}} \xrightarrow{g_{n-1}} \cdots \longrightarrow \overline{F_1}$$

(where the bars represent going modulo c) we can conclude it is acyclic. The long exact sequence on homology associated with the short exact sequence of complexes given by multiplication by c on $\mathbf{F}$, together with Nakayama's lemma, gives the final conclusion. □

3. Uniform Artin–Rees for syzygies

In this section, we take up Question B. By the *rank* of a finitely generated module *M* of finite projective dimension over a local ring, we mean the alternating sum of the ranks of the free modules in a finite free resolution of *M*. Thus, we can ask that a module that is locally of finite projective dimension on the punctured spectrum of a local ring *R* have constant rank on the punctured spectrum. This condition is satisfied, for example, for all modules of finite length (on the punctured spectrum they are free of rank 0!) and all finitely generated modules over an integral domain with an isolated singularity. The goal of this section, and the main result of this paper, is the following:

Theorem 3.1. *Let* $(R, \mathbf{m})$ *be a local Noetherian ring of dimension d, let I be a proper ideal of R, and let*

$$\mathbf{F} : \cdots \longrightarrow F_n \xrightarrow{f_n} F_{n-1} \xrightarrow{f_{n-1}} \cdots \longrightarrow F_0$$

be an exact sequence of finitely generated free modules, which is the resolution of a finitely generated R-module M. Let $B_n = \operatorname{Im}(f_{n+1})$ *denote the submodule of boundaries. If* M_P *has finite projective dimension and constant rank on the punctured spectrum of R, then there exists an integer* $q > 0$ *such that for all n and for all* $N \geqslant q$,

$$I^N F_n \cap B_n \subseteq I^{N-q} B_n.$$

The proof will be given after some preliminary results. If the conclusion of Theorem 3.1 holds for a module *M* then we say that the free resolution of *M* satisfies the uniform Artin–Rees condition with respect to *I*. See Proposition 4.1 for more on the meaning of this condition.

Lemma 3.2. *Let* $(R, \mathbf{m})$ *be a local Noetherian ring, let I be a proper ideal of R, and let*

$$\mathbf{F} : \cdots \longrightarrow F_n \xrightarrow{f_n} F_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} F_0$$

be a complex of finitely generated free R-modules. Set $Z_n = \operatorname{Ker}(f_n)$ *and* $B_n = \operatorname{Im}(f_{n+1})$. *Assume there exists an element* $c \in R$ *such that* $(0 : c)$ *is finite length and such that multiplication by c is homotopic to zero on* $\mathbf{F}_{\geqslant 1}$. *Let an overline denote the natural homomorphism* $R \longrightarrow \overline{R} = R/(c)$. *Assume that* $\mathbf{F}$ *is acylic or, more generally, that there exists an integer l such that for all* $N \geqslant l$ *and all n,*

$$I^N F_n \cap Z_n \subseteq B_n.$$

Let t be an integer such that for all $n \geqslant t$, $I^n \cap (c) \subseteq I^{n-t}c$. *Choose r such that, via the Artin–Rees Lemma,* $\mathbf{m}^r \cap (0 :_R c) = 0$. *Then for all* $N \geqslant \max\{l, r\} + t$ *and for all n,*

$$I^N \overline{F_n} \cap \operatorname{Ker}(\overline{f_n}) \subseteq \overline{B_n}.$$

Proof. Let $u \in I^N F_n$ be chosen so that $\overline{u} \in \operatorname{Ker}(\overline{f_n})$. This means that $f_n(u) = cw$ for some $w \in F_{n-1}$. We have that $cw \in cF_{n-1} \cap I^N F_{n-1} \subseteq cI^{N-t} F_{n-1}$ for $N \geqslant t$. Hence without loss of generality, we may assume that $w \in I^{N-t}F_{n-1}$. In this case, for $N-t \geqslant r$, $f_{n-1}(w)=0$: certainly $cf_{n-1}(w)=f_{n-1}(cw)=f_{n-1}(f_n(u))=0$, and hence

$$f_{n-1}(w) \in (0:_{F_{n-2}}c) \cap I^{N-t}F_{n-2}=0$$

if $N-t \geqslant r$. Thus $w \in Z_{n-1} \cap I^{N-t}F_{n-1} \subseteq B_{n-1}$ for $N-t \geqslant \max\{l,r\}$, in which case we can write $f_n(v)=w$ for some $v \in F_n$, and so $u-cv \in Z_n$.

Let $s_\bullet$ be a homotopy showing multiplication by c on $\mathbf{F}_{\geqslant 1}$ is null-homotopic. Then $c1_{F_n}=f_{n+1}s_n+s_{n-1}f_n$ for all $n \geqslant 1$. In particular, $cv=(f_{n+1}s_n+s_{n-1}f_n)(v) \in I^{N-t}F_n+B_n$ (since $f_n(v)=w$). It follows that $u-cv \in ((B_n+I^{N-t}F_n)+I^NF_n)\cap Z_n=(B_n+I^{N-t}F_n)\cap Z_n=B_n+(I^{N-t}F_n\cap Z_n) \subseteq B_n$ for $N-t \geqslant \max\{l,r\}$. Finally for $N \geqslant \max\{l,r\}+t$, we have that $u=(u-cv)+cv \in B_n+cF_n$ and so $\overline{u} \in \overline{B_n}$. □

The next result provides elements c to which we can apply Lemma 3.2.

Lemma 3.3. *Let $(R,\mathbf{m})$ be a Noetherian local ring of dimension d and let M be a finitely generated R-module such that M_P is free of constant rank over R_P for all primes $P \neq \mathbf{m}$. Assume*

$$\mathbf{F}: \cdots \longrightarrow F_n \xrightarrow{f_n} F_{n-1} \xrightarrow{f_{n-1}} \cdots \longrightarrow F_0$$

is a free resolution of M. There exists a system of parameters $c_1,\ldots,c_d$ such that for all i, multiplication by c_i on $\mathbf{F}_{\geqslant 1}$ is homotopic to 0. Furthermore for all $0 \leqslant i \leqslant d-1$ we can choose c_{i+1} general modulo the ideal $(c_1,\ldots,c_i)$ in the sense that c_{i+1} is contained in no associated prime of $(c_1,\ldots,c_i)$ except possibly the maximal ideal.

Proof. Suppose that $c_1,\ldots,c_i$ have already been chosen satisfying the conclusion of the Lemma. Let $P_1,\ldots,P_k$ be the associated primes of the ideal $(c_1,\ldots,c_i)$, excluding $\mathbf{m}$ if it is associated. By assumption M_{P_j} is free of constant rank for $1 \leqslant j \leqslant k$. Set $W=R-\cup_{j=1}^k P_j$, and $S=R_W$. The module M_W is free over S since it is locally free of constant rank and S is semi-local. Let α be an isomorphism of M_W with F_W for some free module F over R. Let β denote the inverse of α. By clearing denominators, there exists an element $c \in W$ and maps $f: M \longrightarrow F$, $g: F \longrightarrow M$ such that $\alpha=f/c$ and $\beta=g/c$. It follows that after multiplying by a further power of c, which we can relabel as c, $fg=cI_F$ and $gf=I_M$, where I denotes the identity map. This means that the map from M to M induced by multiplication by c factors through a free module $M \xrightarrow{f} F \xrightarrow{g} M$.But then multiplication by c on $\mathbf{F}$ factors through the complex $0 \longrightarrow F \cong F \longrightarrow 0$, which means this multiplication map is homotopic to the zero map on $\mathbf{F}_{\geqslant 1}$. We now set $c_{i+1}=c$. Induction on i completes the proof. □

Proposition 3.4. *Let $(R,\mathbf{m})$ be a local Noetherian ring of dimension d and M a finitely generated R-module such that M_P is free of constant rank for all primes $P \neq \mathbf{m}$. Let*

$$\mathbf{F}: \cdots \longrightarrow F_n \xrightarrow{f_n} F_{n-1} \xrightarrow{f_{n-1}} \cdots \longrightarrow F_0$$

be a free resolution of M. Choose a system of parameters $c_1,\ldots,c_d$ as in Lemma 3.3 and set $R_i=R/(c_1,\ldots,c_i)$ with $R_0=R$. Write F_{ji} (resp. Z_{ji}, B_{ji}) for the free modules $F_j \otimes_R R_i$

(*resp. the cycles, boundaries of* $\mathbf{F}_i := \mathbf{F} \otimes_R R_i$). *Then there exists an integer l such that for all* $i = 1, \ldots, d$, *for all n, and for all* $N \geqslant l$,

$$\mathbf{m}^N F_{ni} \cap Z_{ni} \subseteq B_{ni}.$$

Proof. For $i \geqslant 0$, let t_i be the Artin–Rees number of the embedding $c_{i+1} R_i \subseteq R_i$, that is the least integer such that for all $n \geqslant t_i$, $\mathbf{m}^n R_i \cap c_{i+1} R_i \subseteq \mathbf{m}^{n-t_i} c_{i+1}$. Choose e_i such that $(0 :_{R_i} c_{i+1}) \cap \mathbf{m}^{e_i} R_i = 0$. Set $l_i = t_0 + \cdots + t_{i-1} + e_0 + \cdots + e_{i-1}$. By induction on i we claim that for all $j \leqslant i$, for all n, and for all $N \geqslant l_i$,

$$\mathbf{m}^N F_{nj} \cap Z_{nj} \subseteq B_{nj}.$$

If $i = 0$, there is nothing to prove as the cycles are equal to the boundaries. By induction we can also assume the result for $j \leqslant i - 1$. By the argument above and by Lemma 3.2, we can apply Lemma 3.2 to the ring R_{i-1} and the complex $\mathbf{F}_i$. We apply this with the element c equal to the image of c_i in R_{i-1}. The Artin–Rees number of the embedding $c_i R_{i-1} \subseteq R_{i-1}$, in the above sense, is t_{i-1}. Lemma 3.2 gives us that

$$\mathbf{m}^N F_{ni} \cap Z_{ni} \subseteq B_{ni}$$

for all $N \geqslant t_{i-1} + \max\{e_{i-1}, l_{i-1}\}$. This last integer is at most l_i. □

Proposition 3.5. *Let* $(R, \mathbf{m})$ *be a local Noetherian ring, let I be a proper ideal of R, and let*

$$\mathbf{F} : \cdots \longrightarrow F_n \xrightarrow{f_n} F_{n-1} \xrightarrow{f_{n-1}} \cdots \longrightarrow F_0$$

be a complex of finitely generated free R-modules. Set $Z_n = \mathrm{Ker}(f_n)$ *and* $B_n = \mathrm{Im}(f_{n+1})$. *Assume there exists an element* $c \in R$ *such that* $0 : c$ *is finite length and such that there is a homotopy of the complex showing that multiplication by c is null-homotopic. Let an overline denote the natural homomorphism* $R \longrightarrow \overline{R} = R/(c)$. *Assume that*:

(1) *There exists an integer l such that for all* $N \geqslant l$ *and all n,*

$$I^N \overline{F_n} \cap \mathrm{Ker}(\overline{f_n}) \subseteq I^{N-l} \overline{B_n}.$$

(2) *There is an integer p such that for* $N \geqslant p$ *and for all n,*

$$I^N F_n \cap Z_n \subseteq B_n.$$

Then there exists an integer q such that for all $N \geqslant q$ *and all n,*

$$I^N F_n \cap Z_n \subseteq I^{N-q} B_n.$$

Proof. Let $s_\bullet$ be the homotopy showing multiplication by c is null-homotopic. Then for all n, $c \cdot 1_{F_n} = s_{n-1} f_n + f_{n+1} s_n$. Set $\alpha_n = f_{n+1} s_n$. Then $\alpha_n : F_n \longrightarrow B_n$, and if $u \in Z_n$, then $\alpha_n(u) = (s_{n-1} f_n + f_{n+1} s_n)(u) = cu$.

Let $u \in I^N F_n \cap Z_n$. Note that our assumption forces $u \in B_n$ if $N \geqslant p$, which we will assume. Set $\alpha = \alpha_n$. Then $\alpha : F_n \longrightarrow B_n$ is such that $\alpha(x) = cx$ for all $x \in Z_n$. Applying

α to u, we see that $cu = \alpha(u) \in \alpha(I^N F_n) \subseteq I^N B_n$. Write $cu = \sum_j a_j u_j$ where $u_j \in B_n$ and $a_j \in I^N$. Lift u, u_i back to F_{n+1}, to elements y, y_i, respectively. Set $w = \sum_j a_j y_j \in I^N F_{n+1}$. Then $f_{n+1}(cy - w) = 0$. Going modulo cR, we obtain that

$$\overline{w} \in \operatorname{Ker}(\overline{f_{n+1}}) \cap I^N \overline{F_{n+1}} \subseteq I^{N-l} \overline{B_{n+1}}$$

for all $N \geqslant l$ by our assumption. In particular, the element $w = \sum_j a_j y_j \in I^{N-l} B_{n+1} + cF_{n+1}$. Write $w = z + cv$, where $z \in I^{N-l} B_{n+1}$. Let t be the Artin–Rees number for the embedding $(c) \subseteq R$. Since $w \in I^N F_{n+1}$, it follows that $cv \in cF_{n+1} \cap I^{N-l} F_{n+1} \subseteq cI^{N-l-t} F_{n+1}$ for $N \geqslant l+t$. Hence, because of the expression for w, viz. $w = z + cv$, without loss of generality we may assume that $v \in I^{N-l-t} F_{n+1}$. Apply f_{n+1}. We obtain that

$$cu = f_{n+1}(w) = f_{n+1}(z + cv) = cf_{n+1}(v) \in cI^{N-l-t} B_n.$$

It follows that $u \in I^{N-l-t} B_n + (0 :_{F_n} c)$. Write $u = u' + y'$, where $cy' = 0$ and $u' \in I^{N-l-t} B_n$. By the Artin–Rees lemma and by hypothesis on c, there exists an integer e such that $I^e \cap (0 :_R c) = 0$. Since $u \in I^N F_n$ we obtain that $y' \in (0 :_{F_n} c) \cap I^{N-l-t} F_n = 0$ for $N - l - t \geqslant e$. Then $u = u' \in I^{N-l-t} B_n$. Setting $q = l + t + p + e$ then gives the conclusion. □

Proof of Theorem 3.1. By using the Artin–Rees lemma, we are free to replace **F** by any finite shift of itself. Replacing **F** by $\mathbf{F}_{\geqslant d}$ we can assume that M_P is free of constant rank for all $P \neq \mathbf{m}$. Using Lemma 3.2 choose $c_1, \ldots, c_d$ such that multiplication by each c_i on $\mathbf{F}_{\geqslant 1}$ is homotopic to 0, and such that each c_i is general modulo $(c_1, \ldots, c_{i-1})$.

Set $R_i = R/(c_1, \ldots, c_i)$ with $R_0 = R$. Write F_{ji} (resp. Z_{ji}, B_{ji}) for the free modules $F_j \otimes_R R_i$ (resp. the cycles, boundaries of $\mathbf{F}_i := \mathbf{F} \otimes_R R_i$). Consider the complex $\mathbf{F}_d$. We claim there exists an integer q_d such that for all $N \geqslant q_d$ and for all n,

$$I^N F_{nd} \cap Z_{nd} \subseteq I^{N-q_d} B_{nd}.$$

This follows as I is nilpotent, taking q_d to be the nilpotency index.

We use a backwards induction to prove there exist integers q_i such that for all $N \geqslant q_i$ and all n,

$$I^N F_{ni} \cap Z_{ni} \subseteq I^{N-q_i} B_{ni}.$$

When we reach $i = 0$, we can take $q = q_0$ to finish the proof.

We assume the conclusion for $i+1$. Consider the complex $\mathbf{F}_i$ over the ring R_i. The image of the element c_{i+1} in R_i is general, so that $0 :_{R_i} c_{i+1}$ has finite length (cf. Lemma 3.2). By Proposition 3.4 there is an integer l such that for all $i = 1, \ldots, d$, for all n, and for all $N \geqslant l$,

$$I^N F_{ni} \cap Z_{ni} \subseteq B_{ni}.$$

It follows that the complex $\mathbf{F}_i$ satisfies the conditions of Proposition 3.5, with c_{i+1} playing the role of c in that proposition. The conclusion of Proposition 3.5 proves that there exists an integer q_i such that for all $N \geqslant q_i$ and all n,

$$I^N F_{ni} \cap Z_{ni} \subseteq I^{N-q_i} B_{ni}.$$

The induction finishes the proof. □

4. The meaning of uniform Artin–Rees

The proof of Theorem 3.1 rests heavily on the existence of an **m**-primary ideal J generated by elements c_i with the property that multiplication by c_i on the free resolution of M is homotopic to zero (for a truncated part of the resolution). It follows that J annihilates the functors $\mathrm{Tor}_n^R(M, -)$ for $n \gg 0$. In particular, we can find an integer q such that for all $n \gg 0$ and all $N \geqslant 0$,

$$I^q \,\mathrm{Tor}_n^R(M, R/I^N) = 0.$$

It is natural to inquire how close this 'uniform' annihilation of Tor is to the uniform Artin–Rees statement of Theorem 3.1. In the next proposition we answer this question assuming a small condition on the ideal I, namely that there exists an integer k with $N \geqslant k$ $I^N : I \subseteq I^{N-k}$ for all $N \geqslant K$. This condition is automatic, for example, if I contains a non-zerodivisor x. For in that case, if $uI \subseteq I^N$, then $xu \in (x) \cap I^N \subseteq I^{N-k}x$ by the usual Artin–Rees lemma applied to the modules $(x) \subseteq R$ and the ideal I. Since x is a non-zerodivisor, we may cancel it. The proposition shows that the uniform annihilation of Tor, together with a uniform bound connected with the modules of boundaries are equivalent to the uniform Artin–Rees property for syzygies.

Proposition 4.1. *Let* $(R, \mathbf{m})$ *be a local Noetherian ring, and let I be an ideal of R such that there exists an integer k* with $N \geqslant k$ $I^N : I \subseteq I^{N-k}$ for all $N \geqslant k$. *Let*

$$\mathbf{F} : \cdots \longrightarrow F_n \xrightarrow{f_n} F_{n-1} \xrightarrow{f_{n-1}} \cdots \longrightarrow F_0$$

be an exact sequence of finitely generated free modules, which is the resolution of a finitely generated R-module M. Let $B_n = \mathrm{Im}(f_{n+1})$ *denote the submodule of boundaries. Then the following two statements are equivalent*:

(1) *There exists an integer* $q > 0$ *such that for all* $n \gg 0$ *and for all* $N \geqslant q$,

$$I^N F_n \cap B_n \subseteq I^{N-q} B_n.$$

(2) (a) *There exists an integer* ℓ *such that for all* $n \gg 0$ *and for all* $N \geqslant 0$, $I^\ell \,\mathrm{Tor}_n^R(M, R/I^N) = 0$, *and*
   (b) *there exists an integer m such that for all* $n \gg 0$ *and for all* $N \geqslant m$, $I^N B_n :_{B_n} I \subseteq I^{N-m} B_n$.

Proof. Assume (1). To prove (2a), let u be a cycle representing a class in $\mathrm{Tor}_n^R(M, R/I^N)$. Then $f_n(u) \in I^N F_{n-1} \cap B_{n-1}$, and for $n \gg 0$, by property (1), this latter intersection is contained in $I^{N-q} B_{n-1}$. Hence there exists $a_i \in I^{N-q}$ and $u_i \in F_n$ such that $f_n(u) = \sum_i a_i f_n(u_i)$. Then $u - \sum_i a_i u_i \in B_n$. Multiplying by I^q we obtain that $I^q u \subseteq I^N F_n + B_n$, which implies that I^q annihilates the class of u in $\mathrm{Tor}_n^R(M, R/I^N)$. Thus, we may take $\ell = q$.

To prove (2b), let $u \in B_n$ such that $Iu \subseteq I^N B_n$. Then $Iu \subseteq I^N F_n$, and since F_n is free, the assumptions of the proposition imply that $u \in I^{N-k} F_n \cap B_n \subseteq I^{N-q-k} B_n$ for all $n \gg 0$. Hence we may set $m = q + k$.

Conversely, assume both of the properties of (2). Let $u \in I^N F_n \cap B_n$. Write $u = f_{n+1}(w)$ for some $w \in F_{n+1}$. Then w will represent a class in $\mathrm{Tor}^R_{n+1}(M, R/I^N)$, so that by (2), for $n \geqslant 0$, I^ℓ will annihilate this class, which shows that $I^\ell w \subseteq I^N F_{n+1} + B_{n+1}$. Applying f_{n+1}, we obtain $I^\ell u \subseteq I^N B_n$. By the second statement of (2), this implies that $u \in I^{N-m\ell} B_n$. Taking $q = m\ell$ we get (1). □

5. Elementary examples

To illustrate Question A, we consider the well-known resolution of the residue class field of $R = K[x_1, x_2]/(x_1^2, x_2^3)$, where K is a field. Let $F = R^2(-1)$ be a free module with generators e_1, e_2, and let $E = \wedge F$ be the Koszul complex in which $e_i \mapsto x_i$. It is known from work of Tate [11] that the minimal free resolution of K can be written as a free differential graded divided power algebra $D \otimes E$, where $D = D(G)$ is the divided power algebra on the free module $G = R(-2) \oplus R(-3)$, generated by elements f_2, f_3 of degrees 2 and 3, say, with differential $\mathrm{d} f_2 = x_1 e_1,\ \mathrm{d} f_3 = x_2^2 e_2$. Thus the nth module in the minimal free resolution of K has basis

$$\{f_2^{(\alpha)} f_3^{(\beta)} \otimes e \mid e \in \wedge^t F, 2\alpha + 3\beta + t = n\},$$

with $\mathrm{d}(f_2^{(\alpha)} f_3^{(\beta)} e) = x_1 f_2^{(\alpha-1)} f_3^{(\beta)} e + x_2^2 f_2^{(\alpha)} f_3^{(\beta-1)} e + f_2^{(\alpha)} f_3^{(\beta)} \mathrm{d}(e)$. Here the element $f_2^{(\alpha)} f_3^{(\beta)} \otimes e$ has degree $2\alpha + 3\beta +$ degree e. Since the differential of the Koszul complex E is linear, the given bases allow us to write the differential as a matrix whose entries have degree $\leqslant 2$. On the other hand, if we replace the basis element $f_3^m \otimes 1$ by the element of the same degree $f_3^m \otimes 1 + g f_2^m \otimes 1$, where g is a form of degree m, then the new matrix expressing the differential $F_{2m} \to F_{2m-1}$ has an entry $x_1 g$ of degree $m+1$.

We conclude with an example relevant to the relation of Questions A and B. Consider the graded case. If a map $f : F \to G$ of free modules can be written as a matrix of forms of degree $\leqslant q$ it follows that $\mathbf{m}^N G \cap \mathrm{Im}\, f \subset \mathbf{m}^{N-q}\, \mathrm{Im}\, f$, but the converse is not true, even for the map $f : R \to R^2$ given by the matrix

$$\begin{pmatrix} x \\ y^n \end{pmatrix}$$

over the polynomial ring $K[x, y]$: in this case one sees easily that $\mathbf{m}^N R^2 \cap \mathrm{Im}\, f \subset \mathbf{m}^{N-1}\, \mathrm{Im}\, f$, but f cannot be written as a matrix of lower degree elements.

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