Duality and Socle Generators for Residual Intersections arXiv:1309.2050

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Outline

What is a residual intersection?

Some varieties $X \subset \mathbb{P}^n$ are complete intersections, where $I(X)$ has of codimension s and is generated by s homogeneous forms; then X is (arithmetically) Gorenstein, and Bezout's theorem says (for example) that the degree of X is the product of the degrees of the forms.

Many other varieties can be described as *residual intersections*: roughly, if s hypersurfaces meet in a set X of codimension s AND another variety Y (of any codimension) we say that X is an s-residual intersection of Y. For example, varieties defined by maximal minors of a matrix are usually residual intersections.

Residual Intersections arose in 19th century algebraic geometry in two separate ways, as in the next two examples.

The twisted cubic is the 2-residual intersection of quadrics containing a line

The ideal of the "twisted cubic" curve in $\mathbb{P}^3$,

$$\mathbb{P}^1 \xrightarrow{\cong} C \subset \mathbb{P}^3; \quad (1, t) \mapsto (1, t, t^2, t^3)$$

is generated by 3 quadrics:

$$I(C) = I_2 \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}$$

Thus C is *not* a complete intersection. Any two quadrics containing C, meet in C and a line. For example, the 2×2 minors containing the first column meet in $X = C \cup L$, where L is the line defined by the vanishing of the first column. Thus C is a 2-residual intersection of L ("linked to L").

Definition of residual intersections

The general definition, first studied by Artin and Nagata (1972), takes account of embedded components:

Definition

An ideal K in a Gorenstein ring R is an s-residual intersection of an ideal I if there is an ideal $J \subsetneq I$ generated by s elements such that $K = J : I$ and $\operatorname{codim} K \geq s$.

The case when $s = \operatorname{codim} I$, as in the case of the line and the twisted cubic, is usually called *linkage*; it was important in the 19th century classification of curves in $\mathbb{P}^3$.

Beyond Linkage: How many conics are tangent to 5 given plane conics, $C_1, \ldots, C_5$?

A homogeneous quadratic form in 3 variables has 6 coefficients, so the set of plane conics is naturally $\mathbb{P}^5$. The condition for a conic X to be tangent to a given smooth conic C is the vanishing of a certain form F_C of degree 6 in the coefficients of the equation of X: that is, the set of conics tangent to C is a sextic hypersurface $F_C = 0$ in $\mathbb{P}^5$.

Bézout's Theorem: 5 general sextic hypersurfaces intersect in $6^5 = 7776$ points. *But...*

Theorem (Chasles, 1864)

There are 3264 conics tangent to 5 given general conics: All the F_{C_i} lie in the ideal I of the surface of double lines (the Veronese surface) in $\mathbb{P}^5$; the set of nonsingular conics tangent to 5 given conics is the 5-residual intersection of I.

Singular loci of plane curves

Let $R = \mathbb{C}[x_0, x_1, x_2]$ be the homogeneous coordinate ring of the projective plane, and suppose that $C \subset \mathbb{P}^2_{\mathbb{C}}$ is a plane curve $F = 0$, with isolated singularities. The singular scheme is defined by the saturation I of the ideal J generated by the three partial derivatives of F. What is the significance of I/J as a module over R/K, where $K = J : I$ its annihilator?

Note that K is (the ideal of) a 3-residual intersection of I.

results of van Straten and Huneke

[Recall: $K = J : I \subset R = \mathbb{C}[x_0, \ldots, x_2]$ is a 3-residual intersection.]

About 25 years ago, van Straten showed that I/J is a self-dual module,

$$I/J \cong \mathrm{Hom}_{\mathbb{C}}(I/J, \mathbb{C}) \quad \textit{as modules.}$$

A recent paper of van Straten and Warmt gives a beautiful interpretation of the signature of the resulting quadratic form.

Independently, Huneke and Ulrich realized that the $\mathbb{C}$-dual of R/K, called the *canonical module* of R/K, can be expressed as

$$I^2/JI \cong \mathrm{Hom}_{\mathbb{C}}(R/K, \mathbb{C}).$$

What's the pattern?

Contexts of van Straten, Huneke, Ulrich

- van Straten's result, the self-duality of I/J, means that there is a nonsingular invariant quadratic form on I/J. The duality holds for almost complete intersections J and

$$\frac{(I = \text{ unmixed part of } J)}{J}.$$

- The Huneke-Ulrich result is about the canonical module in a rather general setting, to which we will return.

The condition G_s (Artin-Nagata)

Digression: For $K = J : I$ to be an s-residual intersection, where $J = (f_1, \ldots, f_s)$, the elements $f_1, \ldots, f_s$ must generated I locally at primes of codimension $< s$. Most proofs, at the moment, are inductions on s and require some "general choice of the f_i". To make this work, you need:

Condition G_s: For all $1 \leq j < s$ the ideal I can be generated by $\leq j$ elements at all primes of codimension j.
(Recently: Ulrich, Chardin, Hassanzadeh, Hassanzadeh-R

When is a residual intersection of a Cohen-Macaulay ideal Cohen-Macaulay?

The Cohen-Macaulay condition for residual intersections was studied by Artin-Nagata, Huneke, Ulrich.

Standing Hypothesis $*_s$): Let R be a Gorenstein local ring. Let $I \subset R$ be an ideal of $\operatorname{codim} g$. Suppose that I satisfies G_s and

$$(*) \quad \operatorname{depth}(R/I^{j+1}) \geq \dim(R/I) - j$$

for $j \leq t := s - g$. [True for example for licci ideals]

Theorem (Ulrich)

If I satisfies $*_s$ and $K = J : I$ is an s-residual intersection then K is Cohen-Macaulay of codimension exactly s and $\omega_{R/K} \cong I^{t+1}/JI^t$.

Duality results

Theorem (E-Ulrich)

If I satisfies $*_s$ and $K = J : I$ is an s-residual intersection then the multiplication maps

$$m(I,u,t) : I^u/JI^{u-1} \otimes I^{t+1-u}/JI^{t-u} \xrightarrow{\text{mult}} I^{t+1}/JI^t$$

are perfect pairings.

The case of a geometric s-residual intersection

An s-residual intersection $K = J : I$ is called *geometric* if I and K don't share any components. In this case we can rewrite the previous theorem in a simpler way:

Corollary

Suppose that I satisfies $*_s$ and $K = J : I$ is a geometric s-residual intersection. Let $\bar{I} \subset \overline{R} := R/K$ be the image of I. The truncated Rees algebra

$$\overline{R} \oplus \bar{I} \oplus \bar{I}^2 \oplus \cdots \oplus \bar{I}^{t+1}$$

is Gorenstein: $\bar{I}^{t+1} \cong \omega_{\overline{R}}$ and the multiplication maps $\bar{I}^u \otimes \bar{I}^{t+1-u} \to \bar{I}^{t+1}$ are perfect pairings.

Self-duality of complete intersections

If $K = (f_1, \dots, f_d) \in R = \mathbb{C}[\![x_1, \dots, x_d]\!]$ is a complete intersection, then R/K is self-dual. Equivalently, R/K has a unique minimal ideal, the "socle", isomorphic to the trivial R/K-module $\mathbb{C}$.

Another formulation: The finite-dimensional vector space R/K supports a nonsingular invariant quadratic form, unique up to units.

To determine the quadratic form "on the nose" – for example, in the real case, to determine a signature – we need a canonical generator of the socle ("residue theory").

Jacobian determinant and socle

Theorem

If $K = (f_1, \ldots, f_d) \in R = \mathbb{C}[\![x_1, \ldots, x_d]\!]$ is a complete intersection, then the socle of R/K is generated by the Jacobian determinant of $f_1, \ldots, f_d$:

$$\Delta := \det \begin{pmatrix} \partial f_1/\partial x_1 & \cdots & \partial f_n/\partial x_1 \\ \vdots & \ddots & \vdots \\ \partial f_1/\partial x_n & \cdots & \partial f_n/\partial x_n \end{pmatrix}$$

More generally, if K is a 0-dimensional ideal, then $\omega_{R/K}$ has a 1-dimensional socle, and to define a unique quadratic form on a "middle dimensional" piece we need a canonical generator of the socle.

The case of residual intersections

Theorem (E-Ulrich)

Let k be a field of characteristic 0, and let R be the standard graded polynomial ring in d variables over k. Let $I \subset R$ be a homogeneous ideal and let $J = (f_1, \ldots, f_d) \subset I$, where the f_i all have the same degree, such that $J : I$ is a d-residual intersection.
Suppose that I satisfies $(*_d)$, and in addition is reduced with $\mu(I_P) \leq \operatorname{codim} P - 1$ for all $P \supset I$ with $\operatorname{codim} I < \operatorname{codim} P < d$.

The Jacobian determinant of any d homogeneous generators of J generates the socle of $I^{t+1}/JI^t \cong \omega_{R/K}$.

Remarks: The degree hypothesis is crucial, though we can give a sort of formula in a more general case. However The "additional" hypotheses – reduced with $\mu(I_P) \leq \operatorname{codim} P - 1$ for all $P \supset I$ with $\operatorname{codim} I < \operatorname{codim} P < d$ – seem unnecessary.