Twenty Points in $\mathbf{P}^3$

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Outline

What is Linkage?

The ideal of a rational cubic curve $C \subset \mathbb{P}^3$ is generated by 3 quadrics. If you take just two general quadrics containing C, they define $X = C \cup L$, where L is a secant line of C.

We say that C is *directly linked* to L by the Complete intersection X.

Linkage (or Liaison) is the equivalence class generated by such direct linkages.

History 1

- Linkage (also called *Liaison* has been used a LOT for classifying curves in $\mathbb{P}^3$ since the work of Halphen and Noether (1883).
- Gaeta (1940s) proved that any two arithmetically Cohen-Macaulay curves in $\mathbb{P}^3$ were in the same linkage class—the *linkage class of a complete intersection*. Now one says that they are *licci*.
- The theory was modernized by Peskine–Szpiro.

History 2

- Theorem (Rao): The linkage class of a curve $C \subset \mathbb{P}^3$ is characterized by the finite length graded module

$$H^1_* \mathcal{I}(C) := \oplus_d H^1 \mathcal{I}(C)(d)$$

  up to dualizing and shifting degrees. This extends Gaeta! Schenzel: Linkage "good" in codim 2.
- First salient feature: X of codim 2 is licci iff Arithmetically Cohen-Macaulay.

licci to glicci and the Central Question

- Huneke and Ulrich (and others) showed that in higher codimension, linkage classes were much smaller—no nice classification. And *many* different linkage classes of arithmetically Cohen-Macaulay schemes of codimension 3.
- But: Linkage works with "Gorenstein" in place of complete intersection (Schenzel); and there are a lot more Gorenstein Schemes!
- Central Question: are any two arithmetically Gorenstein Schemes of Codimension 3 "Gorenstein-linked"? That is, is every such scheme in the "Gorenstein linkage class of a complete intersection (*glicci*)??

Hartshorne's Question

Many results show special Arithmetically Cohen-Macaulay schemes are glicci: Kleppe, Migliore, Miró-Roig, Nagel, Peterson, Gorla. For example, determinantal rings, pfaffian rings, Stanley-Reisner rings, ...

but... not the final prize... yet?...

Question(Hartshorne 2001): *What about twenty general points in* $\mathbb{P}^3$*??*

h-Vectors and bi-dominant Gorenstein correspondences

Definintion: If $X \subset \mathbb{P}^n$ is a scheme (say arithmetically Cohen-Macaulay for simplicity) the h-vector of X is the Hilbert function of The homogeneous coordinate ring mod a general linear system of parameters.

Example: If X is a set of $d = \binom{3+m}{3} + k$ general points in $\mathbb{P}^3$, then the h-vector is that of $\mathbb{P}^2$ truncated. For example if $d = 21 = \binom{3+3}{3} + 1$:

$$h = (1, 3, 6, 10, 1, 0, 0, 0 \cdots).$$

On the other hand, the h-vector of an (arithmetically) Gorenstein scheme G is symmetric.

Example: $X = 21$ generic points *could be* directly Gorenstein linked to $Y = 9$ points with general Hilbert function, h-vector $(1, 3, 5)$. The Gorenstein ideal that links them will have h-vector $\{1, 3, 6, 10, 6, 3, 1\}$, and we have:

$$\begin{array}{rlllllll} & 1 & 3 & 6 & 10 & 1 & & \\ + & & & & & 5 & 3 & 1 \\ = & 1 & 3 & 6 & 10 & 6 & 3 & 1 \end{array}$$

Let $Hilb_d^\circ$ be the component of the Hilbert scheme of d-tuples of

Incidence Correspondences

Given d, e and a Gorenstein h-vector of sum $d + e$ and the right form, consider the "incidence correspondence"

$$\Gamma_{d,e,h} \subset Hilb^\circ_d \times Gor_h \times Hilb^\circ_e$$

and the maps

$$Hilb^\circ_d \xleftarrow{\pi_d} \Gamma_{d,e,h} \xrightarrow{\pi_e} Hilb^\circ_e.$$

We say that Γ is *bi-dominant* if π_d and π_e are dominant morphisms.

With luck, it is enough to have one example and check infinitesimally!

There are only finitely many bidominant cases

The bad news: there can only be finitely many (d, e) for which a bidominant Gorenstein linkage correspondence can exist—you can't expect to prove that every set of points in $\mathbb{P}^3$ is glicci so simply.

Reason: $\dim Hilb_d^\circ = 3d$, but for a correspondence we have $\sum h_i = d + e$ and $\dim Gor_h = O((\sum h_i)^{2/3}$.
So: Can list the possibilities. Are they realized?

Gorenstein Schemes containing a set of d general points

- Hartshorne's Refined Question (to Frank): Can you prove that 20 general points are glicci. . . *by computer*?
- Look for pairs: Gorenstein scheme of length $d + e$ containing scheme of length d *defined over a field K where you can compute*.
- Can't *look* over an algebraically closed field K. Can't *find* over $K = \mathbb{Q}$.
- CAN do both over a finite field: Over $K = \mathbb{F}_p$, with p about 10,000, it is an experimental fact that 38% of Gorenstein schemes of h-vector $(1, 3, 6, 10, 6, 3, 1)$ contain a subscheme of length 20 *defined over K!*

Simpler: Rational points on a curve

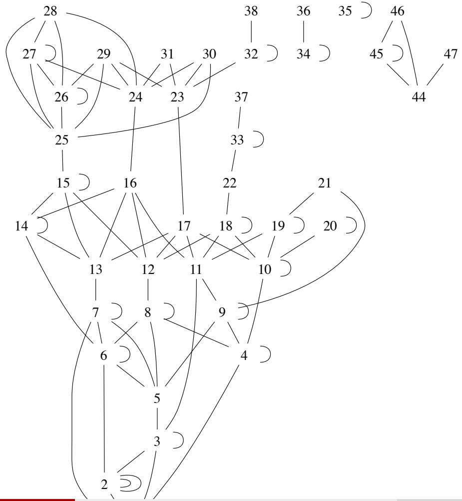
Simplest case of a general principle: Let $C \subset \mathbb{P}^2$ be an irreducible plane curve of degree m over $\mathbb{F}_p$, and let L be a general line in the plane. Then $C \cap L = Spec\ A$ where A is a product of cyclic (Galois) extensions of $\mathbb{F}_p$. The product of the cycles is a permutation of m elements.

Question: Which permutation?

Answer: A *random* one! (—assymptotically for $p \gg 0$.)

In particular, there is a rational point in $L \cap C$ if the permutation has a fixed point; and a subset of d points *defined over* $\mathbb{F}_p$ iff a subset of the cycle lengths add to d. This happens with probability $1 - 1/e \equiv \%63$.

The bidominant Gorenstein linkage correspondences for Hilbert schemes of points in $\mathbb{P}^3_{\mathbb{C}}$ are:



Explanation

An edge *d*—*e* of the graph means that if X is any general set of d points in $\mathbb{P}^3$ then there is a 0-dimensional arithmetically Gorenstein scheme of length $d + e$ containing X; and similarly for general sets of e points.

We say that X is *Gorenstein Linked* to the residual set of e points (in fact G will be reduced—there is nothing "schemey" going on.)

Theorem

Over an algebraically closed field of characteristic zero, a scheme consisting of d general points in $\mathbb{P}^3$ *is glicci when* $d \leq 33$ *and also when* $d = 37$ *or* $d = 38$.

Further cases?

Open question: Is a general set of 34 points in $\mathbb{P}^3$ glicci? A possible attack:

- A general set of 34 points can be linked (using a five-dimensional family of Gorenstein schemes) to a five-dimensional family F of sets of 34 points.
- On the other hand the schemes in $Hilb^{\circ}_{34}$ that are directly Gorenstein linked to 21 points form a subfamily of codimension only 3.
- Hence it is plausible that the family F meets this stratum.
- If this does indeed happen, then, since we know that a set of 21 general points is glicci, it would follow that a set of 34 general points is glicci.