Discussion/Solutions of 2018 Fall *Emissary* puzzles.

Problem 1. Amy and Bob play a game. Initially, there are 20 cookies on a red plate and 14 on a blue plate. A legal move consists either of eating two cookies on one plate, or moving one cookie from the red plate to the blue plate. The last player to move wins, and of course both Amy and Bob want to win. Amy makes the first move, and then they alternate moves. Who will win?

Solution: The game finishes in a finite number of moves since either the total number of cookies or the number of cookies on the red plate decrease with each move. No matter what the starting state, the only two possible end configurations (where no move is possible) are either both plates empty, or the red plate empty and a single cookie on the blue plate. In this specific game, with an initial even number of cookies, the game has to end with both plates empty.

The total number of transfer moves from red to blue must be even, since takeaway moves (eating a pair of cookies) don't change the parity of the number of cookies on the blue plate and the final number is 0. The 34 cookies will be removed by 17 takeaway movies. Therefore the total number of moves will be odd. Since Amy moves first she will win, no matter what strategy (or nonstrategy) she or Bob choose. (Conclusion: the game is less fun to play than it seems at first glance.)

Problem 2. Five people, A, B, C, D, and E, sit around a table playing "Mafia." The players are secretly assigned roles: two are in the mafia, one is a detective, and the other two are innocent (that is, neither a detective nor in the mafia). The mafiosi always lie and the other players always tell the truth. The mafiosi know each other, and the detective knows who they are. The mafia members do not know who the detective is, and the two innocents know nothing about the others' roles. All players know the rules, and the number of players given each role. The following four statements are made in order:

- A: I know who B is.
- B: I know who the detective is.
- C: I know who B is.
- D: I know who E is.

Who is who?

Solution: A's statement can not be made by an innocent (since innocents know nothing initially). So A is a detective or in the mafia. And all players will make this deduction.

B's statement is inconsistent with A being the detective. (If B is innocent then B is not able to make any deductions only knowing A's statement, since A's statement could be made either by a mafiosi or the detective.) So A is in the mafia. This is then clear to all players.

If B is also in the mafia, then A's statement would be true! Since mafiosi never tell the truth, it is clear that B is the detective.

C's statement is true, and C must be an innocent. D knows this, and therefore knows the identity of all players. Since D's statement is true, D is an innocent, and E is the other mafiosi.

Problem 3. Let $A = \{1, 2, 3, 4, 5\}$ and let P be the set of all nonempty subsets of A. A function f from P to A is a "selector" function if $f(B)$ is in B for all nonempty subsets B, and $f(B \cup C)$ is equal either to $f(B)$ or $f(C)$. How many selector functions are there?

Solution: In fact, let A be any finite set, say with n elements. Any total order on A determines a selector function f. Indeed, if S is a nonempty subset, then define $f(S)$ to be the minimum element of S with respect to the given total order. This mapping, from total orders to selector functions, turns out to be a bijection; since total orders on finite sets can be identified with permutations of the set, there are $n!$ selector functions.

To show that the mapping from total orders to selector functions is a bijection, we start by showing that any selector function comes from a total order. Let f be a selector function, and define a relation "$x < y$" on distinct elements of A by saying that $x < y$ if $f(\{x, y\}) = x$. If $x < y$ and $y < z$, i.e., $f(\{x, y\}) = x$ and $f(\{y, z\}) = y$, then

$$\begin{aligned} f(\{x, y, z\}) &= f(\{x, y\}) \text{ or } z \\ &= x \text{ or } z, \end{aligned}$$

and similarly $f(x, y, z) = x$ or y. This implies that $f(\{x, y, z\}) = x$.

Since $f(\{x, y, z\})$ is either y or $f(\{x, z\})$ it follows that $f(\{x, z\}) = x$, i.e., $x < z$.

Therefore this relation is transitive and defines a total order. Moreover, this says that on 3-element subsets S the value $f(S)$ is the minimum element of S. An easy induction argument shows that this is then true of any subset, given that it is true for 3-element subsets. In other words, f comes from the total order. Since two total orders are the same if they agree on all pairwise assignments, f comes from a unique total order, and the association between total orders and selector functions is a bijection.

Problem 4. Let C be a circle of radius ≥ 1 in the plane. You are allowed to augment C by repeatedly performing the following operation: take two points in C that are a distance 1 apart, and add the entire line segment joining them to C. Show that it is possible to perform finitely many such operations so that the center of the circle is in the resulting set.

Solution: Start by taking all pairs of points on the circle that are a distance 1 apart and adding the corresponding line segment to the set. (Yes, that takes infinitely many operations. Hold that thought.) The result is an annulus with outer radius r and inner radius $r' = \sqrt{r^2 - 1/4}$ (from looking at the triangle formed by the endpoints of the line segment and the center of the circle). Continuing in this way gives annuli with inner radii

$$\sqrt{r^2 - 1/4},\ \sqrt{r^2 - 2/4},\ \sqrt{r^2 - 3/4},\ \sqrt{r^2 - 1},\ \sqrt{r^2 - 5/4},\ \ldots\ldots$$

In approximately $2r^2$ steps the inner circle is small enough that the origin lies in the line segment connecting two points in the annulus.

Now work backwards and consider only those points that were ancestors of those two points. Delete all other moves to get a finite set of steps that produces the origin. Each point has two direct ancestors, so this gives an algorithm that takes approximately 2^{2r^2} steps.

Remark: We suspect that there is an $O(r^3)$ algorithm.

Problem 5. A. Assume that a correctly set clock has hour, minute, and second hands of the same length, and that the motion of these hands is continuous. Prove that it is impossible for the tips of those hands to be at the vertices of an equilateral triangle.

Solution: Suppose that the minute hand has done r revolutions since the three hands started together at the high noon position. Here "revolutions" will be taken "modulo 1" since r revolutions and s revolutions are the same for our purposes (i.e., leave the hand in the same positions) if and only if they differ by an integer.

Since the hour hand is slower than the minute hand by a factor of 12, the hour hand is at $r/12$ revolutions and the second hand, being 60 times faster than the minute hand, is at $60r$ revolutions. The three hands, at positions $r/12, r, 60r$, are at the vertices of an equilateral triangle if and only if their differences are $1/3$ modulo 1 (in one cyclic direction or the other), i.e.,

$$(\star) \qquad r - \frac{r}{12} = \pm\frac{1}{3} + m, \quad 60r - r = \pm\frac{1}{3} + n$$

for integers m and n, and where the two signs are the same (with the sign depending on which cyclic order is chosen). Multiplying by 12 and 3 gives

$$11r = \pm 4 + 12m, \qquad 177r = \pm 1 + 3n.$$

Since $11r$ and $177r$ are integers (and 11 and 177 are relatively prime) it follows that r is an integer. The second second equation in $(\star)$ can't be solved by an integer, so it is impossible for the tips of the hands to form an equilateral triangle.

Problem 5. **B**. Suppose that a clock builder chooses a wrong gear and the result is a clock which has correct hour and second hands, but the minute hand makes only 11 revolutions in 12 hours. Show that, despite the clock's various flaws, it does have the virtue that the hand tips do occasionally lie on the vertices of an equilateral triangle. (Assume that the hands start at the high noon position.)

Solution: In this case the speed of the hour hand is only 1/11 of that of the minute hand. The problem is perhaps slightly underspecified in that the speed of the second hand could have (at least) two plausible values, so we will solve the problem that we intended — there are still 60 seconds in a minute — and leave other cases to the reader.

Given this interpretation, after r revolutions of the minute hand, the hands are at $r/11$, r, and $60r$. As above, this gives equations

$$\left(r - \frac{r}{11}\right) = \pm\frac{1}{3} + m, \qquad (60r - r) = \pm\frac{1}{3} + n.$$

The second equation implies that $s := 3r$ is an integer, multiplying by first equation by 33 and the second by 3 gives

$$10s = \pm 11 + 33m, \qquad 59s = \pm 1 + 3n.$$

Either pair of equations can easily be solved in integers, e.g., the equations with the positive sign have solutions of form $s = 11 + 33k$ for arbitrary integers k.

Remark: Larry Carter and Jay-C Reyes came up with both problems. These questions were also used on Stan Wagon's Problem-of-the-week web site in December. One of his readers told him that they were asked the question in part **A** in a physics class in 1964. The questions are capable of innumerable generalizations (different speeds, differing lengths of hands, etc.).

Problem 6. (a) There are n prisoners and $n+1$ hats. Each hat has its own distinctive color. The prisoners are put into a line by their friendly warden, who randomly places hats on each prisoner (one hat is left over). The prisoners "face forward" in line, which means that each prisoner can see all of the hats in front of them. In particular, the prisoner in the back of the line sees all but two of the hats: the one on her own head, and the leftover hat. The prisoners (who know the rules, know all of the hat colors, and have been allowed a strategy session beforehand) must state a guess of their own hat color, in order starting from the back of the line. Guesses are heard by all prisoners. If all guesses are correct, the prisoners are freed. What strategy should the prisoners agree on in their strategy session?

(b) Suppose that there are three prisoners and *five* hats — that is, the warden will place three hats, and there will be two extra hats. The rules are as above. Find an optimal strategy.

Solution: (a): The prisoner at the back of the line sees all but two hats. Given the assumed randomness, that prisoner's statement is a 50/50 coin flip and this puts an upper bound of 50% on the chance that **all** of the prisoners will succeed. Amazingly (at first glance), they can achieve this upper bound by agreeing at their strategy session that they will all behave as if the permutation of the hats is an even permutation. (For this purpose, it might be convenient to imagine that the leftover hat is placed on an imaginary prisoner $n + 1$ who is behind the prisoner at the back of the line, who is neither seen nor heard from.)

Any even permutation is determined if $n - 2$ out of the n elements are known. Therefore the first prisoner, knowing all but two hats, will make a correct statement if (and only if) the permutation is even. The next prisoner knows all hats but his own and the leftover hat (once the first statement is made correctly) and therefore will make a correct statement, under the evenness assumption. And so on down the line.

(b): The first prisoner to state a guess sees 2 colors, and has to choose from among the remaining 3 colors. Therefore the probability of success of any strategy is bounded above by $1/3$. This upper bound can (again, perhaps surprisingly) be achieved. It is easy enough to enumerate strategies and find one with this success probability. One clean way to state an optimal strategy is: the prisoners agree to behave as if their hat colors are in an arithmetic progression modulo 5. In other words, if the hat colors are numbered arbitrarily 1 through 5, and the three prisoner's hat colors are a, b, c, then

$$c - b \equiv b - a \bmod 5, \text{ i.e., } 2b \equiv a + c \bmod 5.$$

It is clear that if the arithmetic progression assumption is true then each prisoner can determine their hat color uniquely from the other two colors.

There are 60 ways that the hats can be placed on the prisoners (i.e., 3-element combinations from 5 things). To verify that the above strategy has success probability $1/3$ we have to show that 20 of those combinations are arithmetic progressions modulo 5. It is not hard to check that starting from single arithmetic progression, say $1, 2, 3$, a total of 20 can be obtained by applying affine transformations $x \to ax + b$ (a and b taken modulo 5, with a nonzero). However, it is also easy just to exhibit the arithmetic progressions: the 10 combinations

$$123, 142, 215, 314, 135, 154, 234, 235, 425, 345,$$

are easily checked to be arithmetic progressions, and their reversals are also obviously arithmetic progressions.

Remark: As shown in the arXiv preprint by Pratt, Wagon, Wiener, and Zielinski (referred to in the original problem statement) if there are two extra hats and at least 7 players, the success probability of the best strategy is strictly less than $1/3$.