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Normal forms for the mode conversion problem

Yves Colin de Verdière
Institut Fourier (Grenoble)
http://www-fourier.ujf-grenoble.fr/~ycolver/

Topics to be discussed

- The mode conversion problem
- What kind of normal forms?
- Genericity hypothesis (topological and symplectic); classification (9 types!): $H_{0,\pm}$, $H_{2,h}$, $H_{2,e}$, $S_{1,h}$, $S_{1,e}$, $H_{P,h}$, $H_{P,e}$, $S_{P,h}$, $S_{P,e}$
- How to get the normal forms?
- Qualitative description of the solutions
- Global problems

What is the mode conversion problem?

We consider a system

$$(\star) \quad \widehat{H}\vec{U} = 0$$

where $\widehat{H} = (\widehat{H}_{i,j})$ is an $N \times N$ self-adjoint matrix of (semi-classical) ΨDO's of order 0 on $\mathbb{R}^d$. The unknown $\vec{U}$ is a map from $\mathbb{R}^d$ into $\mathbb{C}^N$.

The principal symbol H_{class} of $\widehat{H}$ is called the dispersion matrix. It is a map from the phase space $Z = T^\star\mathbb{R}^d$ into the Hermitian $N \times N$ matrices.

The (singular) hypersurface $D = p^{-1}(0)$ where p is the determinant of H_{class} is called the dispersion relation

The kernel L_z of $H_{\text{class}}(z)$ is called the polarisation at z . It is a (singular) bundle. It plays a basic role for WKB solutions:

$$\widehat{H}\left(a(\vec{x})e^{iS(x)/h}\right) = H_{\text{class}}(x, S'(x))\left(a(\vec{x})\right)e^{iS(x)/h} + O(h)$$

The Hamilton-Jacobi equation is $p(x, S'(x)) = 0$.

We want to describe the (micro)local behaviour of the solutions of $(\star)$ using the usual tools: microsupport, Lagrangian states, coherent states, semi-classical measures ...

The situation is well understood near the points where the polarisation bundle is of dimension 1. We have the following reduction tool: let us assume that $\dim P_{z_0} = l$. There exists an unitary ΨDO gauge transform A so that

$$A^\star \widehat{H} A = \begin{pmatrix} \widehat{K} & 0 \\ 0 & \widehat{E} \end{pmatrix}$$

where $\widehat{K}$ is an $l \times l$ system and $\widehat{E}$ is invertible.

In particular if $\dim P_{z_0} = 1$, we get a scalar equation and the usual tools apply: WKB solutions....

We want also to describe the global behaviour: EBK quantization, trace formulae, ...

Mode conversion in physics?

- Propagation of electromagnetic waves in generic media (Maxwell equations, Fresnel sufaces)
- Propagation of acoustic waves in generic media
- Plasma phycics
- Oceanographic waves
- Molecular phycics: Born-Oppenheimer approximation
- Adiabatic Quantum systems (avoided crossings, ...)

Some previous works

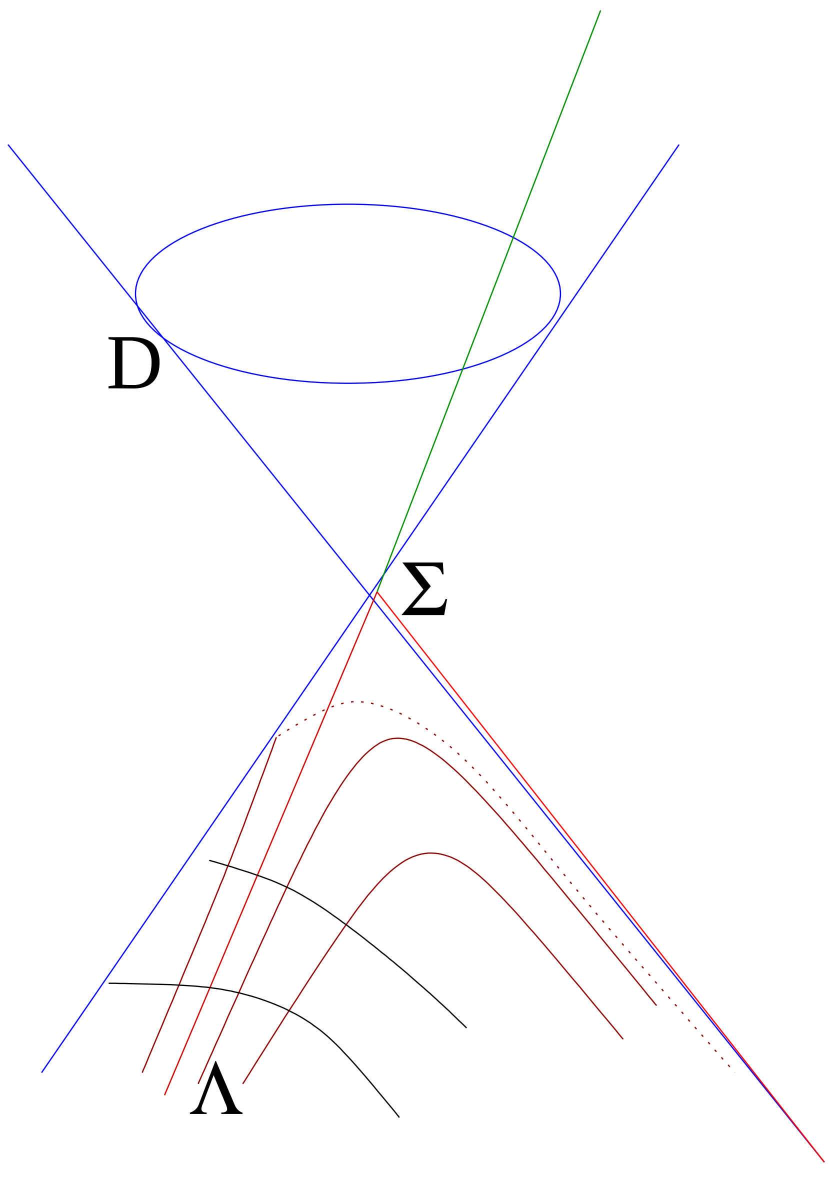
Landau, Zener (1932), Flynn-Littlejohn (1992), Braam-Duistermaat (1993), Hagedorn, Hagedorn-Joye (1994-...), Emmrich-Weinstein (1996), Faure-Zhylinskii (2000-...) Fermanian-Kammerer, P. Gérard, Lasser (2002-..), Tracy-Kaufman (2003), ...

Qualitative propagation

Let us assume that we are in the generic situation: the dispersion relation $D = p^{-1}(0)$ splits into 2 parts: the smooth part where the polarization is a fiber bundle of rank 1 and the singular part Σ where the polarization is of rank > 1.

Let us give a Lagrangian manifold $\Lambda \subset D \backslash \Sigma$, we can associate to it the usual WKB-Maslov states where the amplitude belongs to the polarization bundle. This Λ is invariant by the characteristic flow of p and in general its flow-out will cross Σ.

The main question is: “ What happens there to our solution? “


D
Σ
Λ

Ways to solve the MC problem:

- To find some Ansatz's. We can use the WKB Ansatz outside Σ and linearize the problem near Σ in order to get an approximate solution in some domain which will be smaller as $h \to 0$. This works (Hagedorn) but need some rather difficult analysis with whole pages of terms to estimate!

- The normal form method: here we use much more geometry: using FIO's, we get a (micro)local normal form for which the solutions are easy to compute. We then use families of states (WKB, coherent, ..) whose behaviour w.r. to FIO's is well understood (symbolic calculus)

In the scalar case both methods work as well. It is no more the case in the matrix case.

Normal forms

Everything is (micro-)local near a point z_0 in the phase space with $\dim P_{z_0} = 2$.

We are allowed to use 3 kinds of transformations in order to reduce to a normal form:

1. Reduction to a 2×2 system
2. Scalar FIO's U_χ associated to a canonical transformation χ
3. ΨDO gauge transform $\widehat{A}$ whose principal symbol is a map $A_{\text{class}} : T^\star\mathbb{R}^d \to GL(N, \mathbb{C})$

We get something like:

$$\widehat{A}^{\star} \left(U_{\chi}^{\star} \widehat{H}_{i,j} U_{\chi} \right) \widehat{A} = \widehat{H}_{\mathrm{normal}} \ .$$

At the level of principal symbols, it gives:

$$(A_{\mathrm{class}})^{\star} (H_{\mathrm{class}} \circ \chi) A_{\mathrm{class}} = (H_{\mathrm{normal}})_{\mathrm{class}}$$

And on the level of the dispersion relation:

$$a^2 . p \circ \chi = p_0$$

where we see only the ideals generated by p and p_0.

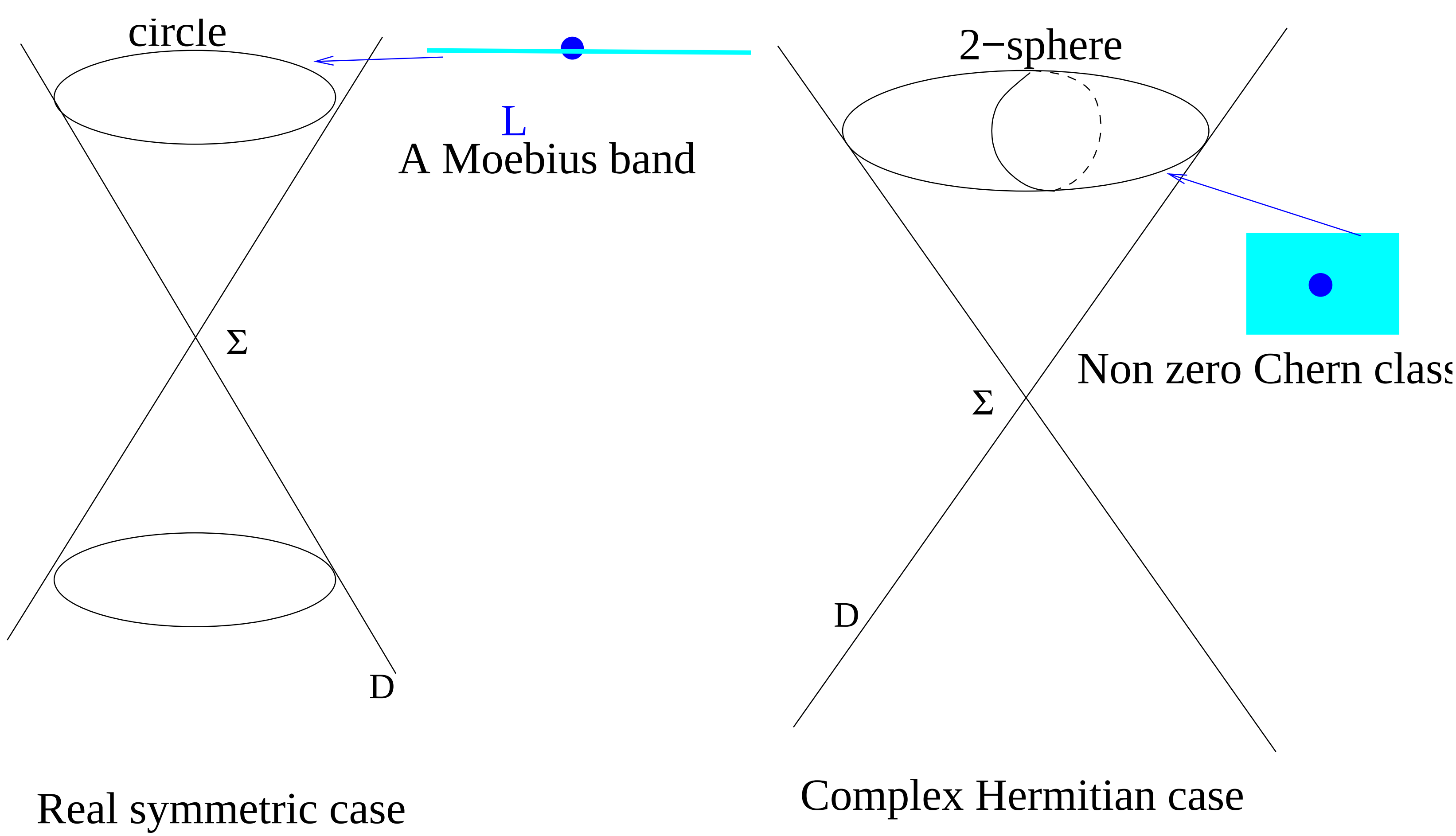
The genericity hypothesis I: topological hypothesis

- Real symmetric case:

  let $W_2^{\mathbb{R}} = \{A \in \mathrm{Sym}(\mathbb{R}^N) | \dim \ker A = 2\}$, we ask that $z \to H_{\mathrm{class}}(z)$ is transversal to W_2. Then $\Sigma = A_{\mathrm{class}}^{-1}(W_2)$ is a codimension 3 submanifold of $T^\star \mathbb{R}^d$ (Wigner-von Neumann)

- Complex Hermitian case:

  let $W_2^{\mathbb{C}} = \{A \in \mathrm{Herm}(\mathbb{C}^N) | \dim \ker A = 2\}$, we ask that $z \to H_{\mathrm{class}}(z)$ is transversal to W_2. Then $\Sigma = A_{\mathrm{class}}^{-1}(W_2)$ is a codimension 4 submanifold of $T^\star \mathbb{R}^d$


circle
L
A Moebius band
2−sphere
Σ
D
Non zero Chern class
Σ
D
Real symmetric case
Complex Hermitian case

The genericity hypothesis II: symplectic hypothesis

They depend only on p; on Σ, we have $p = 0$ and $dp = 0$, so that it makes sense to take the linear part of $\mathcal{X}_p$ on Σ; we will denote it by M.

A. Complex Hermitian case

- H_0: Σ is symplectic. It implies that the eigenvalues of M are $\pm\lambda, \ \pm i\omega$ with $\lambda > 0, \ \omega > 0$.

- H_2: the corank of the restriction of the symplectic form ω to Σ is 2 and M admits one pair of nonzero eigenvalues: this case splits into the elliptic case $H_{2,e}$ and the hyperbolic case $H_{2,h}$. Born-Oppenheimer $H_{\text{class}} = h(x,\xi)\text{Id} + V(x)$ gives $H_{2,h}$ in the generic case.

Analytical form

Let us assume that we have already a 2×2 system.

$$H_{\text{class}} = \begin{pmatrix} p_1 + p_2 & p_3 + ip_4 \\ p_3 - ip_4 & p_1 - p_2 \end{pmatrix} ,$$

we define:

$$\omega_{i,j} = dp_j(\mathcal{X}_i) = \{p_i, p_j\} ,$$

$$\Pi = \omega_{1,2}\omega_{3,4} - \omega_{1,3}\omega_{2,4} + \omega_{1,4}\omega_{2,3}$$

(Π is the Pfaffian of the antisymmetric matrix $(\omega_{i,j})$) and

$$\delta = \frac{1}{8}\text{Tr}(M^2) = \omega_{1,2}^2 + \omega_{1,3}^2 + \omega_{1,4}^2 - \omega_{2,3}^2 - \omega_{2,4}^2 - \omega_{3,4}^2 .$$

We get the following classification:

–The H_0 case corresponds to $\Pi(z_0) \neq 0$. The ratio

$$K := \frac{\omega^2 - \lambda^2}{\lambda\omega} \ ,$$

which is a function of $z' \in \Sigma$, called the Ray Helicity by Tracy and Kaufman, is given by

$$K(z') = -\frac{\delta}{|\Pi|}(z') \ .$$

Chirality: there are in fact 2 non equivalent cases depending on the sign of $\Pi(z_0)$, the $H_{0,+}$ and the $H_{0,-}$ cases.

–The $H_{2,h}$ case corresponds to the vanishing of Π on Σ near z_0 and $\delta(z_0) > 0$

–The $H_{2,e}$ corresponds to the vanishing of Π on Σ near z_0 and $\delta(z_0) < 0$.

B. Real valued symmetric case

We assume that $\omega_{|\Sigma}$ has maximal corank (= 1) and that M admits one pair of nonzero eigenvalues: this case splits into the elliptic case $S_{1,e}$ and the hyperbolic case $S_{1,h}$

C. Systems with parameters

We assume that $d = 1$ for simplicity. We have a system $\widehat{H}_\varepsilon \vec{U} = 0$ where ε is an external parameter. We assume the transversality hypothesis for the mapping $(z, \varepsilon) \rightarrow H_{\text{class}}$ and we assume that $z \rightarrow p_{\varepsilon=0}(z)$ admits a ND critical point at the point z_0. We have again the elliptic and the hyperbolic cases:

$$H_{P,h},\ H_{P,e},\ S_{P,h},\ S_{P,e}$$

Normal forms I: Birkhoff type normal forms for the dispersion relation

In this step, we find χ: it is a normal form problem for a (non generic) scalar Hamiltonian; we use Birkhoff normal form and Sternberg theorem: for example, in the H_0 case, we get

$$p \circ \chi = F(x_1\xi_1, x_2^2 + \xi_2^2, z')$$

which by Taylor formula can be rewrited as:

$$p \circ \chi = a^2 \left(x_1\xi_1 - b(x_2^2 + \xi_2^2, z') \right)$$

Normal forms II: From the dispersion relation to the system

We will use the following result: Let $H : \mathbb{R}^4_X \times \mathbb{R}^N_\lambda \rightarrow \mathrm{Herm}(2)$ be a smooth map such that

$$\det(H(X,\lambda)) = X_1X_2 - (X_3^2 + X_4^2) \ .$$

There exist uniquely defined $\varepsilon = \pm 1$, $\alpha = \pm 1$ and a smooth germ of map $J : \mathbb{R}^4 \times \mathbb{R}^N \rightarrow GL(2,\mathbb{C})$ such that

$$J^\star H(X,\lambda)J = \begin{pmatrix} \alpha X_1 & X_3 + i\varepsilon X_4 \\ X_3 - i\varepsilon X_4 & \alpha X_2 \end{pmatrix}$$

This Lemma is proved using Morse lemma and Moser's path method.

Normal forms III: semi-classics

We will need to solve the following homological equation which is the linearization of:

$$(A_{\text{class}})^\star (H_{\text{class}} \circ \chi) A_{\text{class}} = (H_{\text{normal}})_{\text{class}} ,$$

namely:

$$\{S, H_n\} + B^\star H_n + H_n B = R$$

where R and $H_n = (H_{\text{normal}})_{\text{class}}$ are given and $S,\ B$ are unknown. This equation can be solved for free if the hypothesis on H_{class} are structurally stable.

Normal form for the H_0 case

$$\widehat{H} = \begin{pmatrix} \widehat{\xi_1} & \widehat{B}a \\ a^\star \widehat{B}^\star & x_1 \end{pmatrix} + R$$

where

- $\widehat{B}$ is an elliptic ΨDO whose total symbol is > 0 and depends only on $x_2^2 + \xi_2^2$ and z'
- $a = \widehat{x_2 \pm i\xi_2}$
- The full symbol of R is flat on $x_2 = \xi_2 = 0$

Normal form for the $S_{1,h}$ case

$$\begin{pmatrix} \widehat{\xi_1} & x_2 + ih\widehat{\gamma}(h, \xi_2, z') \\ x_2 - ih\widehat{\gamma}(h, \xi_2, z') & x_1 \end{pmatrix} \vec{U} = 0$$

Normal form for the $H_{P,e}$ case

$$\begin{pmatrix} a_h(\varepsilon) & x_1 + i\widehat{\xi_1} \\ x_1 - i\widehat{\xi_1} & b_h(\varepsilon) \end{pmatrix} \vec{U} = 0$$

The propagation of states in the $S_{1,h}$ case

We start with the simplified normal form:

$$\begin{cases} \frac{h}{i}\frac{\partial u}{\partial x_1} & +x_2 v & =0 \\ x_2 u & +x_1 v & =0 \end{cases}$$

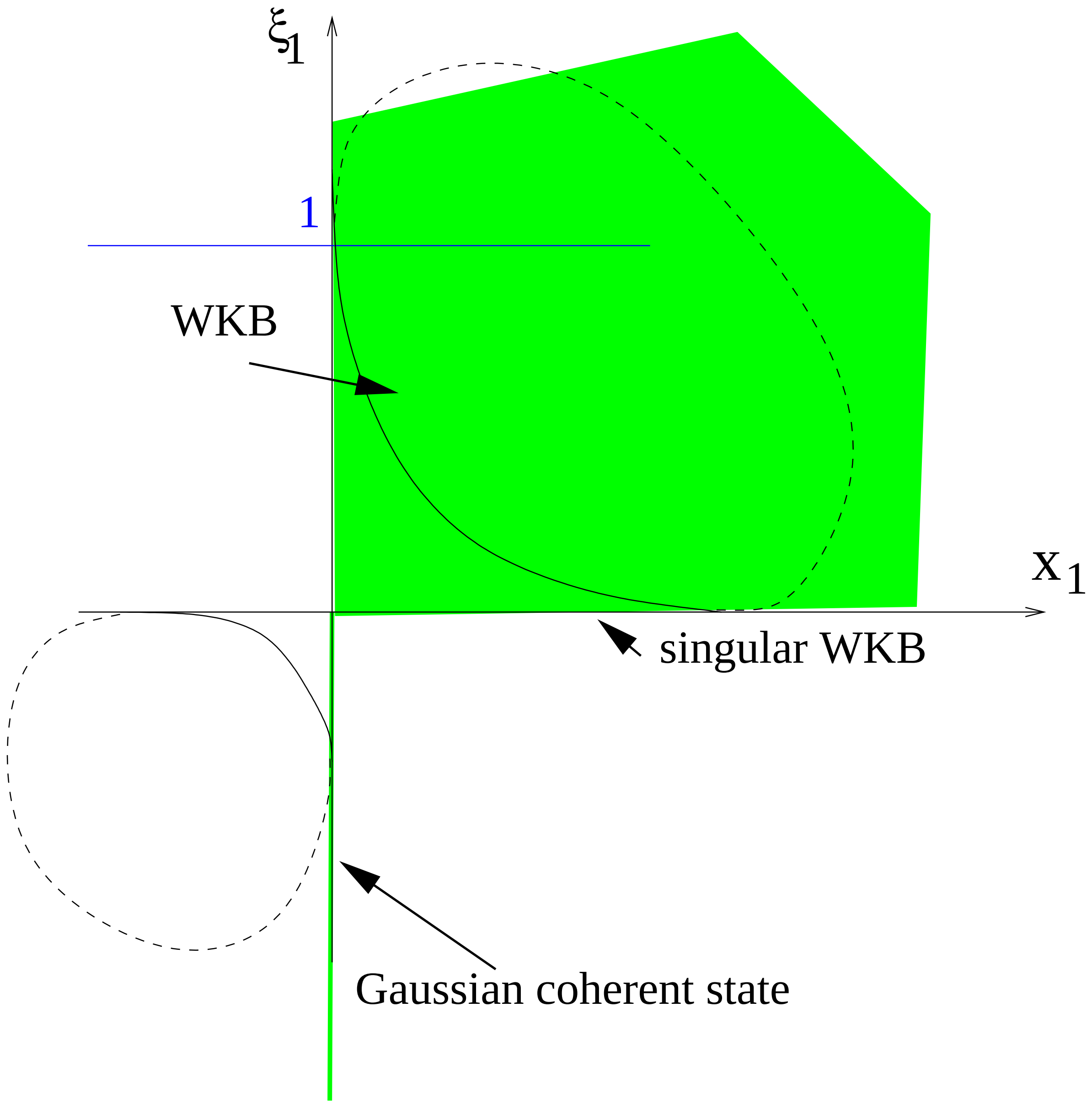
We can easily compute all microlocal solutions. For example solutions supported in $\{x_1 \geq 0\}$ are given by

$$\vec{U} = \varphi_h(x_2, x')\vec{U}_0$$

where

$$\varphi_h(x_2, x') = \widehat{v}(\xi_1 = 1, x_2, x')$$

with where $\widehat{v}$ is h-Fourier transform of v w.r. to x_1.


ξ1
1
WKB
x1
singular WKB
Gaussian coherent state

and $\vec{U}_0$ is given by:

- $x_1 > 0$:

$$\begin{cases} u(x_1,x_2) = -i\sqrt{\frac{2\pi}{h}} Y(x_1) x_2 \left(\Gamma(1 + i\frac{x_2^2}{h})\right)^{-1} e^{\frac{x_2^2}{h}(i \log \frac{x_1}{h} - \frac{\pi}{2})} \\ v(x_1,x_2) = -\frac{x_2}{x_1} u(x_1,x_2) \end{cases}$$

- $\xi_1 > \mathbf{0}$:

$$\begin{cases} \widehat{u}(\xi_1,x_2) = -\frac{x_2}{\xi_1} e^{-\frac{i}{h} x_2^2 \log \xi_1} \\ \widehat{v}(\xi_1,x_2) = e^{-\frac{i}{h} x_2^2 \log \xi_1} \end{cases}$$

- $\xi_1 < \mathbf{0}$:

$$\begin{cases} \widehat{u}(\xi_1, x_2) = \frac{x_2}{|\xi_1|} e^{-\frac{\pi}{h} x_2^2} e^{-\frac{i}{h} x_2^2 \log|\xi_1|} \\ \widehat{v}(\xi_1, x_2) = e^{-\frac{\pi}{h} x_2^2} e^{-\frac{i}{h} x_2^2 \log|\xi_1|} \end{cases}$$

Semi-classical states

- WKB-Maslov states associated to a Lagrangian manifold. Typical form:

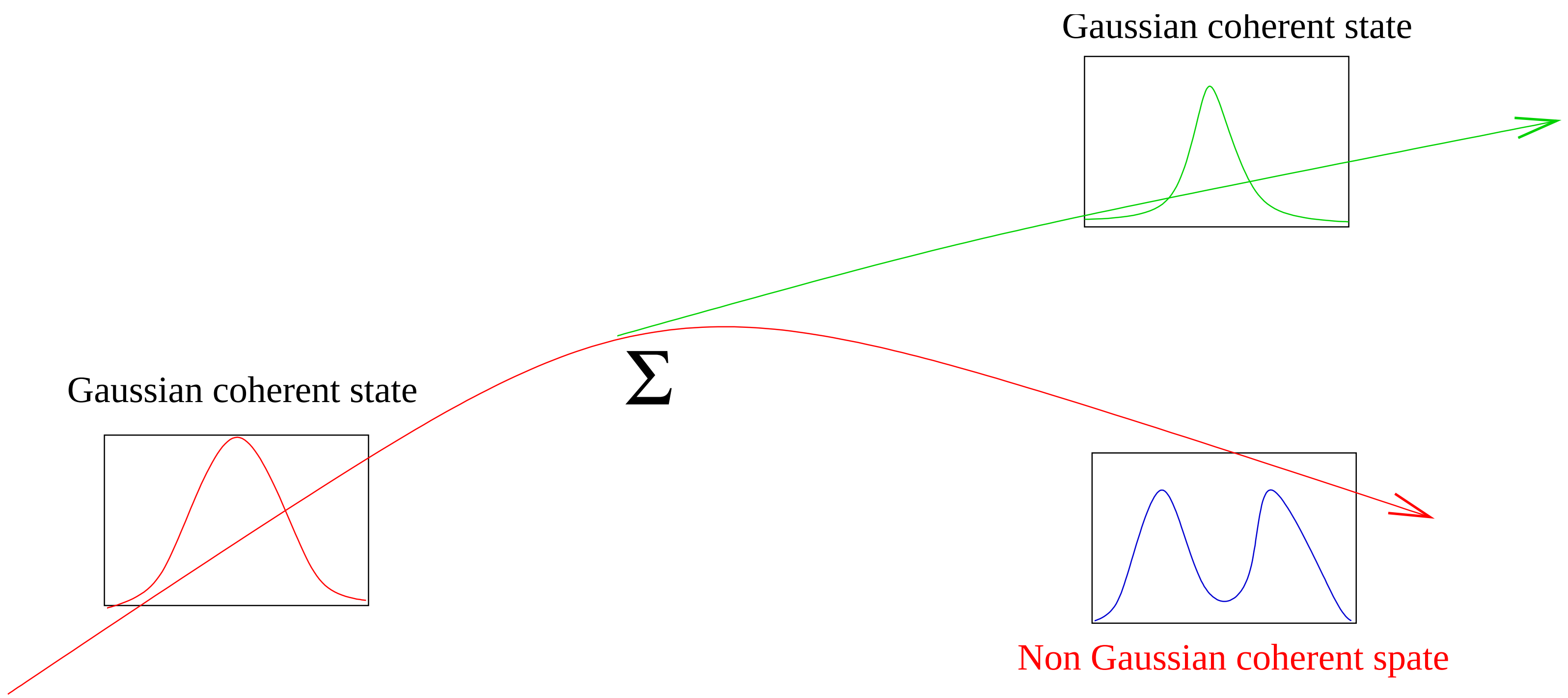
$$a(x)e^{iS(x)/h}$$

- Coherent states also called symplectic spinors associated to an isotropic submanifold. Typical example:

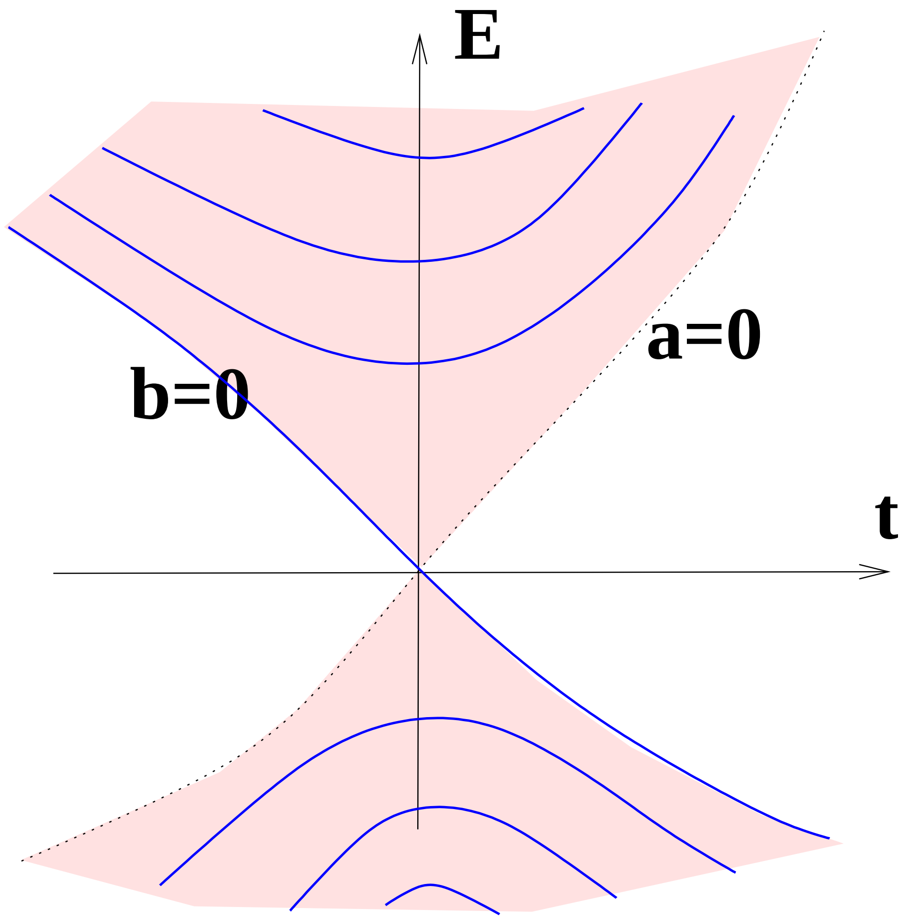
$$a(x, y/\sqrt{h})$$

  with $a(x, Y)$ in the Schwartz space $\mathcal{S}$ w.r. to Y

- Gaussian coherent states. As before, but $a(x, Y)$ is Gaussian w.r. to Y (associated to a complex Lagrangian maifold)


Gaussian coherent state
Gaussian coherent state
Σ
Non Gaussian coherent spate

The propagation of states in the $H_{P,e}$ case



Other singularities can occur:

- Non constant corank of ω_Σ
- Defect of transversality (Σ is singular)
- Bifurcation from the elliptic to the hyperbolic case
- Tripple crossings

It could be interesting to compute the codimensions of these singularities which appear in a stable way if d is bigg enough.

Global problems I: problems in dynamical systems

the dispersion relation is in general an highly non generic Hamilton-Jacobi equation. It would be interesting to know more about the global dynamics of trajectories going near Σ (closed trajectories, ...)

Global problems II: EBK quantization

As suggested by Emmrich-Weinstein, we are interested to describe EBK rules for multicomponent systems. We have to look first for a notion of Quantum Integrability. I suggest the following one (I will restrict myself to 2 degrees of freedom for simplicity): let

$$\widehat{H} = (\widehat{H_{i,j}})$$

be an $N \times N$ Hermitian matrix of ΨDO on $\mathbb{R}^2$. We will say that $\widehat{H}$ is (quantum) integrable if there exists $\widehat{K}$ another Hermitian matrix of ΨDO such that:

$$(\star)\ [\widehat{H}, \widehat{K}] = 0\ .$$

We add as in the scalar case some genericity assumptions for the principal symbols.

Let us assume that we are in some domain of the phase space where the eigenvalue $\lambda(x,\xi)$ of the principal symbol H_{class} is of multiplicity one.

Then because $[H_{\text{class}}, K_{\text{class}}] = 0$, the polarisation bundle $L = \ker(H_{\text{class}} - \lambda)$ is preserved by K_{class}. Let us assume that K_{class} acts on L by multiplication by $\mu(x,\xi)$. Then it is easy to see from $(\star)$ that the Poisson bracket $\{\lambda, \mu\}$ vanishes. Hence we get a scalar integrable system.

Let us fix an invariant Lagrangian manifold Λ of this system. Using both transport equations for a WKB eigenfunction $\vec{a}(x)\text{exp}(iS(x)/h)$ where $\vec{a}(x) \in L_{(x,S'(x))}$, we get a connection on the restriction of L to Λ. From $(\star)$ again we see that this connection is flat. Hence everything reduces to usual BS rules.

Along Σ, both sytems of tori degenerate and we have an interesting bifurcation of EBK rules which could be solved using the tools already developped by Parisse-YC and by San Vũ Ngọc.

A typical example is the following normal form which we have seen before:

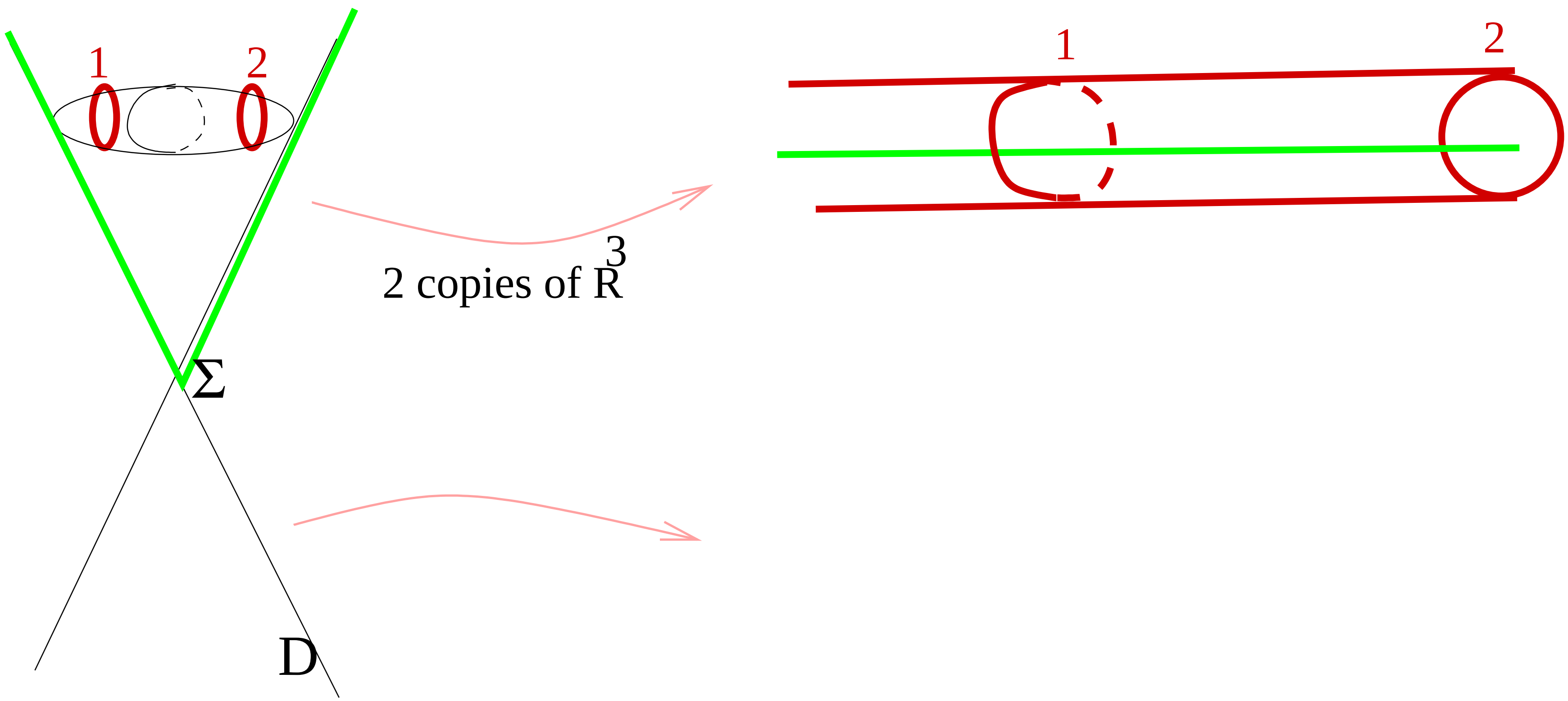
$$\hat{H} = \begin{pmatrix} \widehat{\xi_1} & A \\ A^\star & x_1 \end{pmatrix}$$

with

$$\hat{K} = \begin{pmatrix} AA^\star & 0 \\ 0 & A^\star A \end{pmatrix}$$

This example could be a normal form for the integrable case. It is the case if we restrict to Taylor expansions to order ≤ 2.

Mode conversion in the integrable case



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- Thanks for comments and further discussions