JOINT WORK
WITH DARIO BAMBUSI
MILAN

Uniform Nekhoroshev estimates and Ehrenfest time

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~~Around~~ & ~~??~~ QNF:
SJÖSTRAND - HITRIK - COLIN DE VE..
TOTH ;
EHREN.. BURQ.

HISTORY: QNF ~~PREDATES~~ WAS THERE
BEFORE QM. (BORN - -)
~~IT~~ ITS ROOTS : QUANTIZATION
FORMULAS

1. Classical limit and quantum normal form

Classical limit. $n = (n_1, \ldots, n_l) \in \mathbf{N}^l$ quantum numbers, $J = (J_1, \ldots, J_l) \in \mathbf{R}^d_+$ classical actions. Then:

$$E_n(h) \to E(J) \text{ as } n \to \infty,\ h \to 0, nh \to J \tag{1}$$

Examples

1. $E_n = \sum_{k=1}^{l} (n_l + 1/2) h\omega_k \to \sum_{k=1}^{l} \omega_k J_k.$

2. $E_n = -\dfrac{Z^2}{2(nh)^2} \to -\dfrac{Z^2}{2J^2}.$

Quantization formulas

$$E_n(h) = E(J)|_{J=nh} \qquad \text{(BS)} \tag{2}$$

$$E_n(h) = E(J)|_{J=nh} + hE_1(J)|_{J=nh} + \ldots \tag{3}$$

Computation of E_j: open problem.

Consider the Birkhoff quasi-integrable case:

$$E = \mathcal{H}(x,\xi;\epsilon) = \sum_{k=1}^{l} \omega_k J_k + \epsilon v(x,\xi) \quad (4)$$

$$J_k = \frac{1}{2\omega_k}(\xi_k^2 + \omega_k^2 x_k^2) \quad (5)$$

$$\sum_{k=1}^{l} \omega_k J_k = \frac{1}{2}\sum_{k=1}^{l} (\xi_k^2 + \omega_k^2 x_k^2) := \mathcal{H}_0(x,\xi) \quad (6)$$

Assumptions on v. Define $\Psi : \mathbf{T}^n \times \mathbf{R}^{2l} \to \mathbf{R}^{2l}$

$$(\phi,(\mathbf{x},\xi)) \mapsto (\mathbf{x}',\xi') = \Psi_\phi(\mathbf{x},\xi) \quad (7)$$

$$x_k' := \frac{\xi_k}{\omega_k}\sin\phi_k + x_k \cos\phi_k, \quad (8)$$

$$\xi_k' := \xi_k \cos\phi_k - \omega_k x_k \sin\phi_k \quad (9)$$

Set

$$\tilde{g}_\nu(z) := \frac{1}{(2\pi)^n}\int_{\mathbf{T}^n} g(\Psi_\phi(z)) e^{-i\langle \nu,\phi\rangle}\, d\phi, \qquad k \in \mathbf{Z}^l$$

$$g(\Psi_\phi(z)) = \sum_{k\in\mathbf{Z}^l} \tilde{g}_\nu(z) e^{i\langle k,\phi\rangle} \Longrightarrow g(z) = \sum_{k\in\mathbf{Z}^l} \tilde{g}_\nu(z)$$

g smooth. Let $g \in L^1(\mathbf{R}^{2l})$; set $z := (x,\xi)$

and

$$\widehat{g}(s) := \frac{1}{(2\pi)^{2l}} \int_{\mathbf{R}^{2l}} g(z) e^{-i\langle s,z\rangle}\, dz \quad (10)$$

$$\|g\|_{\rho,\sigma} := \sum_{\nu\in\mathbf{Z}^l} e^{\rho|\nu|} \int_{\mathbf{R}^{2l}} |\widehat{g_\nu}(s)| e^{\sigma\|s\|}\, ds, \rho, \sigma > 0 \quad (11)$$

$$\mathcal{A}_{\rho,\sigma} := \{g : \mathbf{R}^{2l} \to \mathbf{C} : \|g\|_{\rho,\sigma} < +\infty\}.$$

Assumption: $v \in \mathcal{A}_{\rho,\sigma}$. Then:

1.- $f \in \mathcal{A}_{\rho,\sigma}$ entails f analytic on $\mathbf{R}^{2l}$.

2.- Set $V := Op_h^W(v)$ ($h - \psi-$DO of symbol v), i.e.

$$V(u)(x) = \quad (12)$$
$$h^{-d} \int_{\mathbf{R}^l\times\mathbf{R}^l} v((x+y)/2, \xi) e^{i(x-y)\cdot\xi/h} u(y)\, dy\, d\xi$$

$$\|V\|_{L^2\to L^2} \leq \int_{\mathbf{R}^{2l}} |\widehat{v}(s)|\, ds \equiv \|\widehat{v}\|_{L^1} \leq \|v\|_{\rho,\sigma}$$

and the operator family $H_\epsilon = H_0 + \epsilon V$ is self-adjoint with compact resolvents in $L^2(\mathbf{R}^l)$.

3.-Example: $v(x,\xi) = p(x,\xi) e^{-(\|x^2\|+\|\xi\|^2)}$, p polynomial of even degree.

Theorem (Quantum normal form with uniform remainder estimate, [BGP, CMP 1999])

Let ω_k be diophantine, with $\tau > l/2$. Then:
1. $\exists$ analytic bijection $\mathcal{U}(x,\xi;\epsilon,h)$ of $\mathbf{R}^{2l}$ s.t.

$$\mathcal{H}(\epsilon)\circ\mathcal{U} = \mathcal{N}(x,\xi;\epsilon,h) + \mathcal{R}(x,\xi;\epsilon,h) \quad (13)$$

Here $\mathcal{N} := \mathcal{H}_0 + \epsilon\mathcal{Z}(\epsilon,h)$. Moreover

$$\|\mathcal{R}(\epsilon,h)\|_{\rho,\sigma} \le G\exp-(F/\epsilon)^b \quad (14)$$

3. $\mathcal{Z}(x,\xi;\epsilon,h) = \mathcal{Z}(J_1(x,\xi),\dots,J_l(x,\xi);\epsilon,h)$. G,F do not depend on h; $b := 1/(\tau+2)$.
2. $\mathcal{N}$, $\mathcal{Z}$, $\mathcal{R}$, $\mathcal{U}$ are symbols of $h-$ (Weyl) $\psi-$DO $N(\epsilon,h)$, $Z(\epsilon,h)$, $R(\epsilon,h)$, $U(\epsilon,h)$ in L^2. $U(\epsilon,h)$ is unitary and

$$UH(\epsilon)U^{-1} = N(\epsilon,h) + R(\epsilon,h) \quad (15)$$

Here: $N = H_0 + \epsilon Z(\epsilon)$, $[H_0,N] = 0$ and

$$\|R(\epsilon,h)\|_{L^2} \le G\exp[-(F/\epsilon)^b] \quad (16)$$

3. The principal symbols $\mathcal{N}(\epsilon,0)$, $\mathcal{R}(\epsilon,0)$, $\mathcal{U}(\epsilon,0)$ are the Birkhoff normal form, its remainder, and the canonical transformation, *respectively, truncated at a suitable* $K = K(\epsilon)$*, namely*

$$\mathcal{H} \circ \mathcal{U}(\epsilon, 0)(\epsilon, 0) = \mathcal{N}(A_1, \dots, A_l; \epsilon, 0) + \mathcal{R}(x, \xi; \epsilon, 0)$$

Corollary: (Qf with remainder estimate)

$$|E_n(h, \epsilon) - \langle (n + 1/2)h, \omega) \rangle + \epsilon \mathcal{Z}[(n + 1/2)h\omega; \epsilon, h)]| \\ \leq G \exp[-(F/\epsilon)^b]$$

Sketch of the proof. Set

$$U(\epsilon, h) = e^{i\epsilon W(\epsilon,h)/h}, K(\epsilon, h) = U(\epsilon, h) H_\epsilon U(\epsilon, h)^{-1}, \\ W(\epsilon, h) = W_1 + \epsilon W_2 + \dots .$$

Then (PT hierarchy)

$$K(\epsilon, h) = \sum_{k=0}^{\infty} H_k \epsilon^k, \; H_1 = i[H_0, W_1]/h + V$$

$$H_k := \frac{[W_k, P_0]}{ih} + P_k(W_1, \dots, W_{k-1}, H_0)$$

$$\frac{[W_k, H_0]}{ih} + P_k(W_1, \dots, W_{k-1}) = D_k \qquad (17)$$

$$D_0 = H_0, \quad [D_k, H_0] = 0 \qquad (18)$$

Truncating the expansion of W at $k = s-1$:

$$H_s = \sum_{k=0}^{s-1} D_k \epsilon^k + \epsilon^s R_s := N_s(\epsilon) + \epsilon^s R_s \quad (19)$$

$$R_s = R_s(W_1, \dots, W_{s-1}, H_0) \quad (20)$$

To solve, look for semiclassical symbols $\mathcal{W}_k(h)$ (whence $\mathcal{H}_k$, $\mathcal{N}$, $\mathcal{R}$, $\mathcal{U}$). (17) becomes

$$\{\mathcal{W}_k, \mathcal{H}_0\}_M + \mathcal{P}_k(\mathcal{W}_1, \dots, \mathcal{W}_{k-1}, \mathcal{H}_0) = \mathcal{D}_k \quad (21)$$

$$\{\mathcal{D}_k, \mathcal{H}_0\}_M = 0$$

Symbol of $[A(h), B(h)]/ih$:

$$\{\mathcal{A}, \mathcal{B}\}_M(x, \xi) := \mathcal{A} \# \mathcal{B} - \mathcal{B} \# \mathcal{A} \quad (22)$$

$$\{\mathcal{A}, \mathcal{B}\}_M(x, \xi) \sim \{\mathcal{A}, \mathcal{B}\} + \sum_{j=1}^{\infty} \frac{h^{2j}}{2^j} \mathcal{T}_{2j},$$

$$\mathcal{T}_{2j} := \sum_{|\alpha+\beta|=2j+1} \frac{(-1)^{|\beta|}}{\alpha!\beta!} \left\{ \partial_\xi^\alpha D_x^\beta \mathcal{A}, \partial_\xi^\beta D_x^\alpha \mathcal{B} \right\}$$

Crucial remark: $\{\mathcal{A}, \mathcal{H}_0\}_M = \{\mathcal{A}, \mathcal{H}_0\}\ \forall \mathcal{A}$.

(23) becomes a homological equation of CPT:

$$\{\mathcal{W}_k, \mathcal{H}_0\} + \mathcal{P}_k(\mathcal{W}_1, \dots, \mathcal{W}_{k-1}, \mathcal{H}_0) = \mathcal{D}_k \quad (23)$$

$$\{\mathcal{D}_k, \mathcal{H}_0\}_M = 0$$

Chierchia-Gallavotti estimates:

Lemma 1

Let $\mathcal{G} \in \mathcal{A}_{\rho,\sigma}$, $\mathcal{G}' \in \mathcal{A}_{\rho,\sigma-\delta}$. Then $\forall\, 0 < \delta' < \sigma - \delta$:

$$\|\{\mathcal{G}, \mathcal{G}'\}_M\|_{\rho,\sigma-\delta-\delta'} \leq \frac{\|\mathcal{G}\|_{\rho,\sigma}\|\mathcal{G}'\|_{\rho,\sigma-\delta}}{\delta'(\delta'+\delta)e^2}$$

Proof

Weighted L^1 estimates on the $\hat{}$ convolution

$$(\{\mathcal{G}, \mathcal{G}'\}_M)\widehat{}(s) =$$
$$= \frac{2}{h}\int_{\mathbf{R}^{2l}} \widehat{\mathcal{G}}(s^1)\widehat{\mathcal{G}'}(s - s^1)\sin[h(s - s^1) \wedge s^1/2]\, ds^1$$

Lemma 2

If $\mathcal{G} \in \mathcal{A}_{\rho,\sigma}$ the homological equation

$$\{\mathcal{H}_0, \mathcal{W}\}_M + \mathcal{G} = \mathcal{Z}$$

admits analytic solutions $\mathcal{W}$, $\mathcal{Z}$ such that $\mathcal{Z} \circ \Psi_\phi = \mathcal{Z}$, and $\forall\, d < \rho$:

$$\|\mathcal{Z}\|_{\rho,\sigma} \leq \|\mathcal{G}\|_{\rho,\sigma},\ \|\mathcal{W}\|_{\rho-d,\sigma} \leq c d^{-\tau}\|\mathcal{G}\|_{\rho,\sigma}$$

Proof. $\mathcal{H}_0$ is quadratic in $z = (x, \xi)$. Hence

$$\{\mathcal{H}_0, \mathcal{W}\}_M = \{\mathcal{H}_0, \mathcal{W}\} = \frac{d}{dt}\Big|_{t=0} \mathcal{W}(\Psi_{\omega t}(z)) =$$

$$\frac{d}{dt}\Big|_{t=0} \sum_{\nu \neq 0} \frac{\tilde{\mathcal{G}}_\nu(\Psi_{\omega t}(z))}{i\langle \omega, \nu \rangle} = \frac{d}{dt}\Big|_{t=0} \sum_{\nu \neq 0} \frac{\tilde{\mathcal{G}}_\nu(z) e^{i\langle \omega t, \nu \rangle}}{i\langle \omega, \nu \rangle}$$

$$= \sum_{\nu \neq 0} \tilde{\mathcal{G}}_\nu(z)$$ whence

$$||\mathcal{W}||_{\rho - d, \sigma} \leq \sum_{\nu \neq 0} \frac{||\tilde{\mathcal{G}}_\nu||_\sigma}{|\langle \omega, \nu \rangle|} \leq c \frac{||\mathcal{G}||_{\rho, \sigma}}{d^\tau}$$

by the usual small denominators estimates.

Completion of the proof.

Repeated application of Lemmas 1 and 2:

$$||\mathcal{W}_k||_{\rho - (k+1)d, \sigma - (k+1)\delta} \leq \frac{\overline{E} c_1}{d^\tau} \left(\frac{c_2 \epsilon \overline{E}}{\delta^2 d^\tau} \right)^k$$

$$||\mathcal{Z}_k||_{\rho - kd, \sigma - k\delta} \leq \overline{E} \left(\frac{c_2 \epsilon \overline{E}}{\delta^2 d^\tau} \right)^k$$

$$||\mathcal{R}_k||_{\rho - kd, \sigma - k\delta} \leq \overline{E} \epsilon c_3 \left(\frac{c_2 \epsilon \overline{E}}{\delta^2 d^\tau} \right)^k$$

Then choosing $d := \rho/2k$, $\delta := \sigma/2k$ we get

$$\|\mathcal{R}_k\|_{\rho/2,\sigma/2} \leq \overline{E}\epsilon c_3 \left(\frac{c_2\epsilon\overline{E}k^{\tau+2}}{\sigma^2\rho^\tau}\right)^k$$

Finally, truncate at $k(\epsilon) := \left(\frac{\sigma^2\rho^\tau}{\epsilon}\right)^{1/(\tau+2)}$.

2. Quantum Nekhoroshev theorem and Ehrenfest time

In QM, a remainder estimate $\sim e^{-1/\epsilon^b}$ entails quasi-periodicity for $T \sim e^{1/\epsilon^b}$. Here:

Theorem 2 (A uniform quantum Nekhoroshev estimate [BG,2003])

$\exists\, K > 0$, $\Gamma_1 > \Gamma_2 > 0$ independent of h such that, for all $0 \leq t \leq \Gamma_2/\epsilon$:

$$\|B_{N+R}(t) - B_N(t)\| \leq K e^{-(\Gamma_1/\epsilon)^b} \qquad (24)$$

More direct comparison between classical and quantum flows:

Ehrenfest time $T_E(h)$ [Ch,Za]: maximal time interval for which the classical flow determines the quantum one up to $O(h^m)$, $m \geq 1$.

In general, $T_E(h) \sim -\log h$ (proof in[BGP2,BR]). Integrable systems: $T_E(h) \sim h^{-\delta}$, $0 < \delta < 1$? Why this? Semiclassical Egorov theorem: $\mathcal{B}(x,\xi;t)$ symbol of $B(t)$. Then:

$$\mathcal{B}_t(x,\xi;h) \sim (\mathcal{B} \circ \phi_t^{\mathcal{H}})(x,\xi) + \sum_{j=2}^{\infty} \mathcal{B}_j(x,\xi;t)h^j$$

$$\mathcal{B}_j(x,\xi;t) :=$$

$$-i \int_0^t \sum_{\substack{|\alpha+\beta|+l=j+1 \\ 0\le l\le j-1}} \Gamma(\alpha,\beta) \left(\partial_\xi^\alpha \mathcal{H} \partial_x^\beta \mathcal{B}_l\right) \circ \phi_{t-\tau}^{\mathcal{H}}(x,\xi)\, d\tau$$

In general: Lyapunov exponents:

$$\mathcal{B}_t(x,\xi;h) - (\mathcal{B} \circ \phi_t^{\mathcal{H}})(x,\xi) = \sum_{j=2}^{N} \mathcal{B}_j(x,\xi;t)h^j \sim e^{\lambda t}$$

hence $T_E(h) \sim -\log h$ (Proved under restrictive conditions: [BGP2,BR] 99-00). Integrable case: $\lambda = 0$, $T_E(h) \sim h^{-\delta}$.

Remark. As for B_j, construction of N and hence of $e^{iNt/h}$ purely classical. Hence here:

Nekhoroshev estimate (24) = Ehrenfest time

estimate, exponentially long in ϵ *uniformly* with respect to h because:
Ehrenfest time $= +\infty$ for the harmonic oscillators.
More direct comparison between classical and quantum flows: *to what extent and how long $\mathcal{B}(\phi_\tau^{\mathcal{H}}(x,\xi))$ do approximate the (matrix elements of) the Heisenberg observables?*

Let $\mathcal{B} \in \mathcal{A}_{\rho,\sigma}$, and $\mathcal{H}(\epsilon)$ be as above. Define:

$$D_t(B;h,\epsilon) := \sup_{0\leq\tau\leq t;\|\psi\|=1} |\langle\psi, B_\tau\psi\rangle - \mathcal{B}(\phi_\tau^{\mathcal{H}}(x,\xi))| \tag{25}$$

Theorem 3 $\exists\, K_1(B) > 0\,, \Gamma_3(B) > 0$, $\eta(h) = O(h^\infty)$, $\eta(h) > 0$ *such that* $\forall \delta > 0$

$$D_t(B;h,\epsilon) \leq K_1 t^2 \epsilon e^{-(\Gamma_3/\epsilon)^b} + \Gamma_3 h^{1-\delta} t - \eta$$

Remarks

1. Distance (25) defined in Landau-Lifshitz (in action-angle variables, see below).

2. $\epsilon = (-\beta \ln h)^{-1/b}$ yields $D_t(B;h,\epsilon) \sim \Gamma_4 ht-$

η, $T_E(h) \sim h^{-\delta}$ $\forall 0 < \delta < 1$ if $\beta > 2$, even when the perturbation is exponentially large with respect to h.

Proof of Theorem 2. Set:

$$\Delta(t,h) := B_H(t,h) - B_N(t,h)$$

Then one proves the Duhamel formula:

$$\Delta(t,h) = \int_0^t e^{iH(t-s)/h}\frac{[R, B_N(s)]}{ih}e^{-iH(t-s)/h}\,ds$$
$$\Longrightarrow \|\Delta(t,h)\| \le t \max_{0\le s\le t} \|[R, B_N(s)]/ih\| \qquad (26)$$

To estimate the commutator:

Lemma 3. *$\phi^t_{\mathcal{N}}(x,\xi;\epsilon)$ = flow of $\mathcal{N}(\epsilon)$. Let:*

$$r^t_1 := \Delta_{\mathcal{N}}(\mathcal{B}\circ\phi^t_{\mathcal{N}}) = \frac{1}{h^2}[\{\mathcal{B}\circ\phi^t_{\mathcal{N}},\mathcal{N}\}-\{\mathcal{B}\circ\phi^t_{\mathcal{N}},\mathcal{N}\}_M]$$

Then, $\forall 0 < \delta < \sigma$, $\mathcal{B}\circ\phi^t_{\mathcal{N}}, r^t_1 \in \mathcal{A}_{\rho,\sigma-\delta}$ if $|t| < \Gamma_2/\epsilon$.

Proof: write

$$r^t_1 = \int_{\mathbf{R}^{2n}} (\hat{\mathcal{B}}\circ\phi^t_{\mathcal{N}})(s^1)\hat{\mathcal{Z}}(s-s^1) \times$$

$$\times \left\{ (s - s^1) \wedge s^1 - \frac{2}{h} \sin[h(s - s^1) \wedge s^1/2] \right\} \frac{1}{h^2} ds^1$$

By Lemma1, for $|t| < \Gamma_2/\epsilon$:

$$\|r_1^t\|_{\mathcal{A}_{\rho,\sigma-\delta}} \le \frac{2}{e^2\delta} \|\mathcal{Z}\|_{\mathcal{A}_{\rho,\sigma}} \cdot \|\mathcal{B} \circ \phi^t_{\mathcal{N}}\|_{\mathcal{A}_{\rho,\sigma}} \quad (27)$$

Lemma 4 $\exists \Gamma_3 = \Gamma_3(B) > 0$ such that

$$\|i[R, B_N(t)]/h\|_{L^2 \to L^2} \le \Gamma_3 t \epsilon e^{-(\Gamma_2/\epsilon)^b} \quad (28)$$

Proof. Semiclassical Egorov theorem with remainder ([BGP2], Lemma 2.1)

$$B_N(t) = Op^W(\mathcal{B} \circ \phi^t_{\mathcal{N}}(x,\xi)) - h^2\Theta(t) (29)$$

$$\Theta(t) := \int_0^t e^{i\frac{H\tau}{h}} Op^W(r^1(t-\tau)) e^{-i\frac{H\tau}{h}} d\tau (30)$$

Therefore:

$$\|i\frac{[R, h^2\Theta(t)]}{h}\| \le 2ht\frac{\epsilon e^{-(\Gamma_2/\epsilon)^b}}{e^2\delta} \|\mathcal{Z}\|_{\mathcal{A}_{\rho,\sigma}} \cdot \|\mathcal{B} \circ \phi^t_{\mathcal{N}}\|_{\mathcal{A}_{\rho,\sigma}}$$

by (27) and $\|F\|_{L^2 \to L^2} \le \|\mathcal{F}\|_{\rho,\sigma}.\forall\ \rho > 0$, $\sigma > 0$. Moreover, by Lemmas 3, 1:

$$\|i\frac{[R, Op^W(\mathcal{B} \circ \phi^t_{\mathcal{N}})]}{h}\| \le$$

$$\frac{1}{2e^2\delta^2} \|\mathcal{R}\|_{\rho,\sigma-\delta} \|\mathcal{B} \circ \phi^t_{\mathcal{N}}\|_{\rho,\sigma-\delta} \le \frac{\epsilon e^{-(\Gamma_2/\epsilon)^b}}{2e^4\delta^5} \|\mathcal{B} \circ \phi^t_{\mathcal{N}}\|_{\rho,\sigma}$$

whence (28).

Now by (26,28) $\exists \Gamma_1 = \Gamma_1(B) > 0$ such that

$$\|\Delta(t,h)\| \le \Gamma_1 t^2 \epsilon e^{-(\Gamma_2/\epsilon)^b}$$

Proof of Theorem 3

e_k: $k = (k_1, \ldots, k_l)$ (normalized) eigenvectors of H_0. Let $\psi = \sum_k a_k e_k$. By Thm. 2:

$$\sup_{\|\psi\|=1} |\langle \psi, e^{i(N+R)t/h} B e^{-i(N+R)t/h} \psi \rangle - \langle \psi, e^{iNt/h} B e^{-iNt/h} \psi \rangle| \le \Gamma_1 t^2 \epsilon e^{-(\Gamma_2/\epsilon)^b} \quad (31)$$

Want to estimate

$$|\langle \psi, e^{iNt/h} B e^{-iNt/h} \psi \rangle - \mathcal{B} \circ \phi_t^{\mathcal{N}}(x,\xi)| \quad (32)$$

In the the action-angle variables

$$J_l = \frac{1}{2\omega}(\xi_l^2 + \omega_l^2 x_l^2), \quad \theta_l = -\text{Arctg}\left(\frac{\xi_l}{\omega_l x_l}\right) (33)$$

$$x_l = \sqrt{\frac{2J}{\omega}} \cos\theta, \quad \xi_l = -\sqrt{2\omega J} \sin\theta_l \quad (34)$$

$\mathcal{B}(x(J;\theta),\xi(J,\theta)) = \mathcal{B}(\mathcal{J},\theta)$ and

$$\mathcal{B}(J,\theta) = \sum_{\nu\in\mathbf{Z}^l} \mathcal{B}_\nu(J)e^{i\langle\nu,\theta\rangle}, \qquad \nu = (\nu_1,\dots,\nu_l)$$

$\mathcal{B} \in \mathcal{A}_{\rho,\sigma}$ $\exists \Gamma_5 > 0$ s.t.

$$\sup_{J\in\mathbf{R}^l} |\mathcal{B}_\nu(J)| \le e^{-\Gamma_5\|\nu\|} \tag{35}$$

$$\phi_t^{\mathcal{N}}(J,\theta) := \{J(t) = J; \theta(t) = \omega(J)t + \theta\}, \omega := \nabla_J\mathcal{N}$$

$$(\mathcal{B}\circ\phi_t^{\mathcal{N}})(J,\theta) = \sum_{\nu\in\mathbf{Z}^l} \mathcal{B}_\nu(J)e^{i\langle\nu,\theta\rangle + i\langle\omega(J),\nu\rangle t}$$

Lemma 5. $\forall M > 0$ $\exists$ $\Gamma_6(M) > 0$ *such that:*

$$|\langle e_k, B\mathbf{e}_s\rangle| \le e^{-\Gamma_6\|\nu\|},\ \|k-s\| > M \tag{36}$$

Lemma 6 As $k\to\infty$, $h\to 0$, $kh \to J$:

$$\langle \mathbf{e}_k, B\mathbf{e}_l\rangle = \sum_{r=0}^{\infty} \mathcal{B}^s(J,\nu)h^r;\ \mathcal{B}^0(J,\nu) = \mathcal{B}_\nu(\mathcal{J})$$

For any $\Omega \subset\subset \mathbf{R}^n$ $\exists$ $M_r(\Omega) > 0$ such that

$$\max_{J\in\Omega} |\mathcal{B}_\nu^r(J)| \le \|\nu\|^r e^{-M_r\|\nu\|} \tag{37}$$

Choose now $a_k = h^{l/2}f(kh)$,

$$\psi = \sum_{k_1,\dots,k_n} h^{n/2} f(k_1h,\dots,k_lh)e_{k_1,\dots,k_l} \tag{38}$$

$f(x) : \mathbf{R}^l \to \mathbf{R}$ smooth and normalized

$$\|\psi\|_{L^2}^2 = \sum_{k \in \mathbf{Z}^l} h^l |f(kh)|^2 \to \int_{\mathbf{R}^l} |f(x)|^2 \, dx_1 \dots dx_l = 1$$

$$\sum_k h^n |\nabla f(k_1 + \alpha_1 h, \dots, k_n h + \alpha_n h)|^2 \to$$

$$\to \int_{\mathbf{R}^n} |\nabla f(x_1, \dots, x_n)|^2 \, dx_1 \dots dx_n < +\infty \quad (39)$$

Now compute, setting $s = k + \nu$, and

$$e^{i\mathcal{N}(k,h,t)} := e^{i\{\mathcal{N}[(k+1/2)h] - \mathcal{N}[(k+\nu+1/2)h]\}/h}$$

$$\langle \psi, e^{iNt/h}\psi \rangle = \sum_{k,|\nu| \le M} a_k \bar{a}_{k+\nu} B_{k,k+\nu} e^{i\mathcal{N}(k,h,t)} +$$

$$\sum_{k,|\nu|>M} a_k \bar{a}_{k+\nu} B_{l,k+\nu} e^{i\mathcal{N}(k,h,t)} := I_1(B,\psi) + I_2(B,\psi)$$

Now, by (36):

$$|I_2(B,\psi)| \le \sum_{k \in \mathbf{Z}^l} |a_k|^2 \sum_{\|\nu\|>M} |e^{-\Gamma_6 \|\nu\|}$$

$$\le 2e^{-\Gamma_6 M} \frac{e^{-\Gamma_6}}{1 - e^{-\Gamma_6}} = \Gamma_7 e^{-\Gamma_6 M}, \ \Gamma_7 > 0 \quad (40)$$

Now $I_1(B,\psi)$. $\|\nu\|$ is bounded. Expanding

$$\{\mathcal{N}[(k+1/2)h] - \mathcal{N}[(k+\nu+1/2)h,]\}/h\} =$$

$$\langle \nabla_J \mathcal{N}((k+1/2)h), \nu\rangle t + \frac{1}{2} ht \langle \text{Hess}\mathcal{N}(\mathcal{J})_{\mathcal{J}=\mathcal{J}^*}\nu, \nu\rangle$$

$$= \langle \omega(k_1+1/2)h, \nu\rangle t + \frac{1}{2} ht \langle \text{Hess}\mathcal{N}(\mathcal{J})_{\mathcal{J}=\mathcal{J}^*}\nu, \nu\rangle$$

Expand ω in powers of h around $J = kh$, and the resulting exponential in powers of ht. The result is:

$$e^{i\mathcal{N}(k,h,t)} =$$

$$= [1 + \frac{1}{2} ht \langle \mathcal{Q}(J)\nu, \nu\rangle + O(\|\nu^2\| h^2 t^2)] \exp i \langle \omega(J), \nu\rangle t$$

$$\mathcal{Q}(J)_{r,s} := \text{Hess}\mathcal{N}(J)|_{J=J^*} + \text{Hess}\mathcal{N}(J)|_{J=k_l}$$

Apply Lemma 6 to $B_{k,k+\nu}$. By (37) we get:

$$B_{k,k+\nu} = \mathcal{B}_\nu(J_1, \ldots, J_n)|_{J_l=k_l h} + O(h) \quad (41)$$

Moreover, expanding and applying (39):

$$a_{k+\nu} = f(kh) + h\langle \nabla_J f(J), \nu\rangle|_{J=(k+\alpha)h}, \ |\alpha_i| < \nu_i$$

$$a_k \overline{a}_{k+\nu} = h^n |f(kh)|^2 + O(h) \qquad (42)$$

$$I_1(B, \psi) = \sum_{k \in \mathbf{Z}^l} h^n |f(kh)|^2 [1 + Mh] \times$$

$$\times \sum_{\|\nu\| \leq M} [\mathcal{B}_\nu(kh)(1 + O(h))] \times$$

$$\times[1+\frac{1}{2}ht\langle \mathcal{Q}(J)\nu,\nu\rangle + O(\|\nu^2\|h^2t^2)]\exp i\langle\omega(J),\nu\rangle t$$

Now choose $M = h^{-\eta}$ $\forall\eta > 0$; let $h \to 0$, $kh = J$.

$$|I_1(B,\psi) - \sum_{\nu\in\mathbf{Z}^l} \mathcal{B}_\nu(J)\exp i\langle\omega(J),\nu\rangle t| < \Gamma_5 h^{1-\eta}t,$$

If $M = h^{-\eta}$, then by (40) $|I_2(B,\psi)| = O(h^\infty)$.

The proof is complete on account of (31).