CLOSED G_2-STRUCTURES

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1. Some G_2 linear algebra. Recall that G_2 is the subgroup of $GL(7,\mathbb{R})$ that preserves the 3-form

$$\phi_0 = dx^{123} + dx^{145} + dx^{167} + dx^{246} - dx^{257} - dx^{356} - dx^{347},$$

where dx^{ijk} means $dx^i{\wedge}dx^j{\wedge}dx^k$.

The group G_2 is a compact, 1-connected subgroup of $SO(7)$ that acts transitively on $S^6 \subset \mathbb{R}^7$, with isotropy group isomorphic to $SU(3)$ (in its standard representation on $\mathbb{R}^6$).

Since $\mathbb{R}^7$ is odd dimensional, a maximal torus in G_2 must leave a vector in $\mathbb{R}^7$ fixed and is therefore conjugate to a torus in an isotropy subgroup, i.e., $SU(3)$. Since $SU(3)$ has rank 2, it follows that G_2 also has rank 2 and that a maximal torus for $SU(3)$ is also a maximal torus for G_2. Though $SU(3)$ has a center isomorphic to $\mathbb{Z}_3$, these central elements fix a vector in $\mathbb{R}^7$ and are therefore not central in G_2.

Thus, G_2 has trivial center, so that all of its nontrivial representations are faithful.

Representations: The representation ring of G_2 is generated by its two fundamental representations:

The first is $V_{1,0} \simeq \mathbb{R}^7$. The second is $V_{0,1} \simeq \mathfrak{g}_2$, which has dimension 14.

The first few remaining representations are given in the following table, where the subscript is the highest weight vector and the superscript is the (real) dimension.

$V^{1}_{0,0}$	$V^{14}_{0,1}$	$V^{77}_{0,2}$
$V^{7}_{1,0}$	$V^{64}_{1,1}$	$V^{286}_{1,2}$
$V^{27}_{2,0}$	$V^{189}_{2,1}$	$V^{729}_{2,2}$
$V^{77}_{3,0}$	$V^{448}_{3,1}$	$V^{1547}_{3,2}$

For $p \geq 0$, the representation $V_{p,0}$ is isomorphic to $S^p_0(\mathbb{R}^7)$, the harmonic polynomials on $\mathbb{R}^7$ of degree p and $K(\mathfrak{g}_2) \simeq V_{0,2}$.

Exterior Algebra. The G_2-irreducible decompositions of the vector spaces $\Lambda^p(\mathbb{R}^7)$ will be important.

Of course $\Lambda^1(\mathbb{R}^7)$ and $\Lambda^6(\mathbb{R}^7)$ are isomorphic to $\mathbb{R}^7$ and so are irreducible. However, $\Lambda^p(\mathbb{R}^7)$ for $1 < p < 6$ are reducible.

By duality, it suffices to describe the decompositions of $\Lambda^2(\mathbb{R}^7)$ and $\Lambda^3(\mathbb{R}^7)$.

The decomposition of $\Lambda^2(\mathbb{R}^7)$ follows from the embedding of G_2 into SO(7):

$$\Lambda^2(\mathbb{R}^7) \simeq \mathfrak{so}(7) \simeq \mathfrak{g}_2 \oplus \mathfrak{g}_2{}^{\perp} \simeq \mathfrak{g}_2 \oplus \mathbb{R}^7,$$

so we write $\Lambda^2(\mathbb{R}^7) \simeq \Lambda^2_{14}(\mathbb{R}^7) \oplus \Lambda^2_7(\mathbb{R}^7)$.

These summands can be described explicitly as

$$\begin{aligned}\Lambda^2_7(\mathbb{R}^7) &= \{\, *_0(\alpha \wedge *_0\,\phi_0) \mid \alpha \in \Lambda^1(\mathbb{R}^7)\,\} \\ &= \{\, \alpha \in \Lambda^2(\mathbb{R}^7) \mid \alpha \wedge \phi_0 = 2 *_0 \alpha \,\} \\ \Lambda^2_{14}(\mathbb{R}^7) &= \{\, \alpha \in \Lambda^2(\mathbb{R}^7) \mid \alpha \wedge \phi_0 = - *_0 \alpha \,\}\end{aligned}$$

Similarly, there is an irreducible decomposition

$$\Lambda^3(\mathbb{R}^7) \simeq \Lambda^3_{27}(\mathbb{R}^7) \oplus \Lambda^3_7(\mathbb{R}^7) \oplus \Lambda^3_1(\mathbb{R}^7),$$

where these summands have the explicit descriptions

$$\begin{aligned}\Lambda^3_1(\mathbb{R}^7) &= \{\, r\phi_0 \mid r \in \mathbb{R}\,\} \\ \Lambda^3_7(\mathbb{R}^7) &= \{\, *_0(\alpha \wedge \phi_0) \mid \alpha \in \Lambda^1(\mathbb{R}^7)\,\} \\ \Lambda^3_{27}(\mathbb{R}^7) &= \{\, \alpha \in \Lambda^3(\mathbb{R}^7) \mid \alpha \wedge \phi_0 = \alpha \wedge *_0\,\phi_0 = 0 \,\} \simeq S^2_0(\mathbb{R}^7)\end{aligned}$$

Explicitly, define $\mathsf{i} : S^2(\mathbb{R}^7) \to \Lambda^3(\mathbb{R}^7)$ by

$$\mathsf{i}(\alpha \circ \beta) = \alpha \wedge *_0\,(\beta \wedge *_0\,\phi_0) + \beta \wedge *_0\,(\alpha \wedge *_0\,\phi_0).$$

Then $\mathsf{i}\big(S^2_0(\mathbb{R}^7)\big) = \Lambda^3_{27}(\mathbb{R}^7)$. Defining $\mathsf{j} : \Lambda^3(\mathbb{R}^7) \to S^2(\mathbb{R}^7)$ by

$$\mathsf{j}(\gamma)(v,w) = *_0\big((v \lrcorner \phi_0) \wedge (w \lrcorner \phi_0) \wedge \gamma\big),$$

for $\gamma \in \Lambda^3(\mathbb{R}^7)$ and $v, w \in \mathbb{R}^7$, one finds that

$$\mathsf{j}\big(\mathsf{i}(h)\big) = -8h$$

for all $h \in S^2_0(\mathbb{R}^7)$.

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2. G_2 Structures. Let M^7 be a smooth 7-manifold. Recall that a G_2-structure on M is a 3-form ϕ on M such that, for each point $x \in M$, there exists an isomorphism $u : T_xM \to \mathbb{R}^7$ such that $u^*(\phi_0) = \phi_x$.

M possesses a G_2-structure iff M is orientable and spinnable. The set of G_2-structures on M will be denoted $\Omega^3_+(M) \subset \Omega^3(M)$. These 3-forms are the sections of an open subbundle $\Lambda^3_+(T^*M)$ of $\Lambda^3(T^*M)$.

Each $\phi \in \Omega^3_+(M)$ has an associated Riemannian metric g_ϕ and orientation $*_\phi 1 \in \Omega^7(M)$.

Given a G_2-structure $\phi \in \Omega^3_+(M)$, the G_2-equivariant decompositions of $\Lambda^p(\mathbb{R}^7)$ induce corresponding decompositions of $\Omega^p(M)$. For example,

$$\begin{aligned}\Omega^2_7(M,\phi) &= \{\, \beta \in \Omega^2(M) \mid \beta \wedge \phi = 2 *_\phi \beta \,\} \\ \Omega^2_{14}(M,\phi) &= \{\, \beta \in \Omega^2(M) \mid \beta \wedge \sigma = - *_\phi \beta \,\}.\end{aligned}$$

Recall the theorem of Fernandez and Gray:

Theorem: Let σ be a G_2-structure on M. Then σ is parallel with respect to its associated metric g_σ if and only if $\mathrm{d}\sigma = \mathrm{d}(*_\sigma \sigma) = 0$.

There is a general formula for the derivatives of a G_2-structure:

Proposition: For any G_2-structure $\sigma \in \Omega^3_+(M)$, there exist unique differential forms $\tau_0 \in \Omega^0(M)$, $\tau_1 \in \Omega^1(M)$, $\tau_2 \in \Omega^2_{14}(M,\sigma)$, and $\tau_3 \in \Omega^3_{27}(M,\sigma)$ so that the following equations hold:

$$\begin{aligned} d\,\sigma &= \tau_0 *_\sigma\sigma + 3\,\tau_1 \wedge \sigma + *_\sigma \tau_3\,, \\ d *_\sigma \sigma &= \qquad 4\,\tau_1 \wedge *_\sigma\sigma + \tau_2 \wedge \sigma\,. \end{aligned}$$

Remarks: Except for the appearance of τ_1 in two places, this follows directly from the σ-decomposition of exterior forms.

For any $G \subset SO(n)$, the torsion of a G-structure on M^n takes values in a bundle modeled on $(\mathfrak{so}(n)/\,\mathfrak{g}) \otimes \mathbb{R}^n$. In our case:

$$\big(\mathfrak{so}(7)/\,\mathfrak{g}_2\big) \otimes \mathbb{R}^7 \simeq V_{1,0} \otimes V_{1,0} \simeq V_{0,0} \oplus V_{1,0} \oplus V_{0,1} \oplus V_{2,0}$$

essentially by dimension count.

Recall that $K(\mathfrak{g}_2) \simeq V_{0,2} \simeq \mathbb{R}^{77}$, which implies Bonan's result that a metric with holonomy in G_2 must be Ricci-flat.

It follows that, for the general G_2-structure, it must be possible to express the Ricci tensor in terms of the torsion forms τ_0, τ_1, τ_2 and τ_3. The result (got by routine calculation) is:

Proposition For any G_2-structure $\sigma \in \Omega^3(M)$, the following hold:

$$\mathrm{Scal}(g_\sigma) = 12\,\delta_\sigma\tau_1 + \tfrac{21}{8}\,{\tau_0}^2 + 30\,|\tau_1|^2 - \tfrac{1}{2}\,|\tau_2|^2 - \tfrac{1}{2}\,|\tau_3|^2.$$

and

$$\begin{aligned} \mathrm{Ric}(g_\sigma) = &-\Big(\tfrac{3}{2}\,\delta\tau_1 - \tfrac{3}{8}\,{\tau_0}^2 + 15\,|\tau_1|^2 - \tfrac{1}{4}\,|\tau_2|^2 + \tfrac{1}{2}\,|\tau_3|^2\Big)\,g_\sigma \\ &+ \mathrm{j}\Big(-\tfrac{5}{4}\,d\big(*_\sigma(\tau_1 \wedge *_\sigma\sigma)\big) - \tfrac{1}{4}\,d\tau_2 + \tfrac{1}{4}\,*_\sigma d\tau_3 \\ &\quad + \tfrac{5}{2}\,\tau_1 \wedge *_\sigma(\tau_1 \wedge *_\sigma\sigma) - \tfrac{1}{8}\,\tau_0\tau_3 + \tfrac{1}{4}\,\tau_1 \wedge \tau_2 \\ &\quad + \tfrac{3}{4}\,*_\sigma(\tau_1 \wedge \tau_3) + \tfrac{1}{8}\,*_\sigma(\tau_2 \wedge \tau_2) + \tfrac{1}{64}\,\mathsf{Q}(\tau_3,\tau_3)\Big)\,. \end{aligned}$$

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3. Closed G_2 Structures. From now on, I will only be considering G_2-structures $\sigma \in \Omega^3_+(M)$ that are closed, i.e., $d\sigma = 0$.

By the previous formulae, it follows that, for such a structure, one has $\tau_0 = \tau_1 = \tau_3 = 0$ and

$$d{*_\sigma}\sigma = \tau_2 \wedge \sigma$$

where τ_2 lies in $\Omega^2_{14}(M,\sigma)$. In particular,

$$\tau_2 \wedge *_\sigma\sigma = 0 \qquad \text{and} \qquad \tau_2 \wedge \sigma = -*_\sigma\tau_2\,.$$

The Ricci and scalar curvature formulae simplify to

$$\mathrm{Scal}(g_\sigma) = -\tfrac{1}{2}\,|\tau_2|^2$$

and

$$\mathrm{Ric}(g_\sigma) = \tfrac{1}{4}\,|\tau_2|^2\,g_\sigma - \tfrac{1}{4}\,\mathrm{j}\left(d\tau_2 - \tfrac{1}{2}\,*_\sigma(\tau_2 \wedge \tau_2)\right).$$

In particular, note that the scalar curvature is pointwise non-positive and vanishes identically if and only if σ is also coclosed.

These formulae show that g_σ is Einstein if and only if

$$\mathrm{d}\tau_2 = \tfrac{3}{14}|\tau_2|^2\,\sigma + \tfrac{1}{2}*_\sigma(\tau_2\wedge\tau_2)$$

Using this, Cleyton and Ivanov (`math.DG/0306362`) have recently shown that any closed G_2-structure on a *compact* manifold whose associated metric is Einstein must actually be co-closed as well.

Hitchin's volume functional and flow. Suppose that M^7 is compact and let $S \in H^3_{\mathrm{dR}}(M,\mathbb{R})$ be a cohomology class. Define

$$\mathcal{Z}_+(S) = \{\ \sigma \in \Omega^3_+(M) \mid \mathrm{d}\sigma = 0,\ [\sigma] = S\ \}$$

as the set of closed G_2-structures whose de Rham cohomology class is S. Note that $\mathcal{Z}_+(S)$ is an open subset of S (which is an affine subspace of $\mathcal{Z}^3(M)$, the space of closed 3-forms on M).

Hitchin defined a function $f : \mathcal{Z}_+(S) \to \mathbb{R}^+$ by

$$f(\sigma) = \int_M *_\sigma 1 = \int_M \sigma\wedge *_\sigma\sigma.$$

Proposition: (Hitchin) $\sigma \in \mathcal{Z}_+(S)$ is a critical point of f if and only if $\mathrm{d} *_\sigma\sigma = 0$. All the critical points of f are nondegenerate, modulo the action of the diffeomorphism group. The gradient flow of the functional f is given by

$$\frac{d}{dt}(\sigma) = \Delta_\sigma\sigma = \mathrm{d}(\delta_\sigma\sigma)$$

Steve Altschuler and I had considered this (transversely) parabolic flow in 1992 with an eye towards trying to construct compact manifolds with holonomy G_2. Here are some of the results that we derived about it.

From now on, write τ instead of τ_2, for simplicity. We have

$$\mathrm{d}\sigma = 0, \qquad \text{and} \qquad \mathrm{d}*_\sigma\sigma = \tau\wedge\sigma$$

where $*_\sigma(\tau\wedge\sigma) = -\tau$. One can now easily compute that

$$d\tau = \tfrac{1}{7}|\tau|^2\,\sigma + \gamma$$

for some $\gamma \in \Omega^3_{27}(M,\sigma)$.

The evolution equation (Hitchin's flow) becomes

$$\frac{d}{dt}(\sigma) = \mathrm{d}\tau.$$

The formulae from the previous page then imply

$$\frac{d}{dt}(*_\sigma 1) = \tfrac{1}{3}|\tau|^2 *_\sigma 1.$$

Note that the volume form is increasing *pointwise*, and not just on average (as would be expected for the f-gradient flow). Thus,

$$\frac{d}{dt}\big(f(\sigma(t))\big) = \frac{1}{3}\int_M |\tau|^2 *_\sigma 1$$

Computing a further derivative and integrating by parts yields the formula

$$\frac{d^2}{dt^2}\big(f(\sigma(t))\big) = \frac{1}{3}\frac{d}{dt}\int_M |\tau^2| *_\sigma 1 = \int_M \left(\tfrac{2}{9}|\tau|^4 - \tfrac{2}{3}|\mathrm{d}\tau|^2_\sigma\right) *_\sigma 1$$