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Outline:

1) cross product structures
2) G_2-structures — compared with Kähler
3) Decomposition of Λ^*
4) Deformations —
   1) conformal
   2) infiniesimal - by a V.F.
   3) non-infinitesimal by a V.F.

1 on $\mathbb{R}^n$, $\langle\,,\,\rangle$

$$X : \underbrace{\mathbb{R}^n \times \dots \times \mathbb{R}^n}_{k} \to \mathbb{R}^n,$$ k-linear and alternating

is a <u>cross product</u> if

(i) $\langle X(u_1, \dots, u_k), u_i \rangle = 0 \quad \forall i = 1, \dots k$

(ii) $|X(u_1, \dots, u_k)|^2 = |u_1 \wedge \dots \wedge u_k|^2$

Note: from X you get a $(k+1)$-fo[rm] α:

$\alpha(u_1, \dots, u_{k+1}) = \langle X(u_1, \dots, u_k), u_{k+1} \rangle$

Classified by Brown & Gray (1967)

(i) $k = n-1$

$X(u_1, \dots, u_{n-1}) = *(u_1 \wedge \dots \wedge u_n)$ $\qquad \alpha$ = volume form

(ii) $n = 2m$ $\quad k = 1$ $\quad J : \mathbb{R}^{2m} \circlearrowleft$, $J^2 = -I$, an a.c. str., $\alpha = \omega$ kähler form

(iii) $n = 7$ $\quad k = 2$ $\quad G_2$ structure $\quad \alpha = \varphi$, 3-form

(iv) $n = 8$ $\quad k = 3$ $\quad$ Spin$_7$ structure $\alpha = \Phi$, 4-form

Define: M 7-mfd

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G_2-structure on M is a ~~globally defined, smoothly~~ smooth 3-form which is everywhere non degenerate.

This defines a cross-product and metric globally.

In $\mathbb{R}^7$:

$$u \times (u \times v) = -|u|^2 v + \langle u, v \rangle u$$

If φ is a 3-form defining a G_2-structure

$$(v \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \varphi = -6 |v|^2 \, \mathrm{vol}$$

this is the definition of non-degeneracy

$$\langle u \times v, w \rangle = \varphi(u, v, w)$$

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Kahler J	**G_2** $\times$
$\omega(u,v) = \langle Ju, v\rangle$	$\varphi(u,v,w) = \langle u \times v, w\rangle$
g & J are independent	φ (or $\times$) determines g
The manifold is Kähler	The manifold is a G_2-manifold
$\iff \nabla J = 0$	$\iff \nabla \times = 0$
$\iff \nabla \omega = 0$	$\iff \nabla \varphi = 0$
$\iff \mathrm{Hol} \subset U(m)$	$\iff \mathrm{Hol} \subset G_2$
Equivalently: J integrable & $d\omega = 0$	Equivalently $d\varphi = 0$, $d *_\varphi \varphi = 0$ ← non-linear PDE since $*$ depends on φ.
Not all Ricci Flat	All are Ricci Flat

Kähler $\rightarrow$ Kähler + Ricci $=0$ To obtain holonomy $\subset SU(n)$

$U(n) \rightarrow SU(n)$

1 dim

(Thm) Calabi-Yau: Given a compact, simply connected Kähler manifold (M, ω)
1978

$\exists!$ Ricci-flat Kähler metric in the same

Kähler class $\Leftrightarrow c_1 = 0$

Idea of Pf: Use $\partial\bar{\partial}$-lemma: $\tilde{\omega} = \omega + \partial\bar{\partial} f \xrightarrow{\text{get}}$ an elliptic PDE for f. (complex Monge-Ampère equation)

Then prove existence and uniqueness of solutions.
This is a hard analysis problem.

To get $\mathrm{Hol} \subset G_2$, no intermediate starting pt.

$$SO(7) \rightarrow G_2$$

$21 \longrightarrow 14$ "expect" a PDE for a V.F. or a 1-form

difference is 7 dimensions

Here there is no "$\partial\bar{\partial}^*$-lemma".

Let φ_0 be a fixed G_2-structure.

Given φ_0, consider $\tilde{\varphi} = \varphi_0 + \eta$ for some 3-form η.

We want to study when $\tilde{\varphi}$ is a new G_2-structure and what it's properties are in terms of φ_0.

③ Decomposition of Λ^* for a G_2 structure (M, φ_0)

Λ^k_ℓ = ℓ-dim'l subspace of Λ^k, irreducible representation of G_2

This is the G_2-analogue of the Hodge diamond.

$$
\begin{array}{cccc}
\Lambda^7_1 & & & \\
 & \Lambda^6_7 & & \\
 & \Lambda^5_7 & \Lambda^5_{14} & \\
\Lambda^4_1 & \Lambda^4_7 & & \Lambda^4_{27} \\
\Lambda^3_1 & \Lambda^3_7 & & \Lambda^3_{27} \\
 & \Lambda^2_7 & \Lambda^2_{14} & \\
 & \Lambda^1_7 & & \\
\Lambda^0_1 & & &
\end{array}
$$

Subspaces in same vertical column are all naturally isomorphic.

For example,

$$\Lambda^3_1 = \{ f\varphi_0 \, , \, f \in C^\infty(M) \}$$

$$\Lambda^3_7 = \{ w \lrcorner * \varphi \, , \, w \in \Gamma(TM) \}$$

$$\Lambda^3_{27} = \left\{ \eta \in \Lambda^3 \quad \begin{array}{l} \varphi \wedge \eta = 0 \\ *\varphi \wedge \eta = 0 \end{array} \right\}$$

This decomposition will change as φ changes, unlike the Kähler case where the metric can change without changing the Hodge structure. (Since J and g are independent in Kähler case.)

Simplest possibility: $\tilde{\varphi} = f^3 \varphi_0$ $\quad f$ non-vanishing (Gray)

$$\Rightarrow \tilde{g} = f^2 g_0$$

Fernandez
Cabrera
Ugarte

$$d\varphi \in \Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{27}$$
$$d*\varphi \in \Lambda^5_7 \oplus \Lambda^5_{14}$$

$\tau_1 \in \Lambda^1_7$ is isomorphic to component of $d\varphi$ in Λ^4_7 $\cong$ component of $d*\varphi$ in Λ^5_7

Conformally can eliminate τ_1.

Now consider <u>infinitesimal</u> deformations in the Λ^3_7 - direction:

consider φ_t s.t.

$$\frac{d}{dt}\varphi_t = W \lrcorner *_t \varphi_t \quad \text{for fixed } W \in \Gamma(TM)$$

it looks nonlinear but in fact, under this flow the metric doesn't change, $g_t = g_0 = \text{const.}$

so $*_t = *_0$

so $$\frac{d}{dt}\varphi_t = W \lrcorner *_0 \varphi_t$$

The solution is $$\varphi_t = \varphi_0 + \frac{\left(1-\cos(|W|t)\right)}{|W|^2}\left(W \lrcorner * (W \lrcorner * \varphi_0)\right)$$
$$+ \frac{\sin(|W|t)}{|W|}(W \lrcorner * \varphi_0)$$

This yields, for each choice of vector field w, a closed path of pos. 3-forms, all with same metric. | Implicit in Bryant-Salomon 89

Special Case:

i) $N = CY$ 3-fold (Ω, ω)

$M = N \times S^1_\theta$ $\quad \varphi = \mathrm{Re}(\Omega) - d\theta \wedge \omega$ is a G_2-structure.

Let $w = \frac{\partial}{\partial\theta}$ then

$$\varphi_t = \mathrm{Re}(\underbrace{e^{it}\Omega}) - d\theta \wedge \omega$$

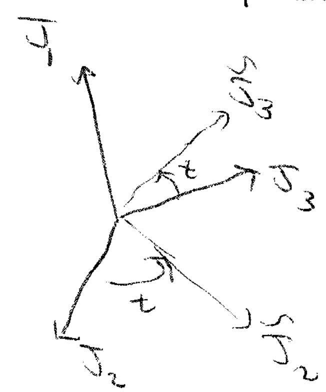
this is the phase freedom for the holomorphic $(n,0)$-form.

ii) $L = K3$ $\quad \omega_1, \omega_2, \omega_3$ (Hyperkähler ~~...~~ $\dim_{\mathbb{C}} = 2$)

$M = L \times T^3$ $\quad \varphi = d\theta_1 \wedge d\theta_2 \wedge d\theta_3 - \sum_{i=1}^{3} d\theta_i \wedge \omega_i$ is a G_2-structure

$$\varphi_t = d\theta^1 \wedge d\theta^2 \wedge d\theta^3 - d\theta^1 \wedge \omega_1 - d\theta^2 \wedge \left[(\cos t)\,\omega_2 + \sin(t)\,\omega_3\right]$$
$$- d\theta^3 \wedge \left[(-\sin t)\,\omega_2 + (\cos t)\,\omega_3\right]$$

This is a hyper-Kähler rotation by angle t around J_1 axis.



Try non-infinitesimal v.f. deformation: (7)

$$W \in \Gamma(TM)$$

$$\tilde{\varphi} = \varphi_0 + \underbrace{W \lrcorner *_0 \varphi_0}_{\Lambda^3_7}$$

a priori, this may not be a G_2 structure.

Thm: (K-) $\tilde{\varphi}$ is still a positive 3-form

The new metric $\langle\ ,\ \rangle_\sim$ is

$$\langle V_1, V_2 \rangle_\sim = \frac{1}{(1+|W|_0^2)^{\frac{1}{3}}}\left(\langle V_1, V_2\rangle_0 + \langle W\times V_1, W\times V_2\rangle_0 \right)$$

with old cross product

$$= \frac{1}{(1+|W|_0^2)^{\frac{1}{3}}}\left[\langle V_1,V_2\rangle_0 + |W|_0^2 \langle V_1, V_2\rangle_0 - \langle W, V_1\rangle_0 \langle W, V_2\rangle_0 \right]$$

Geometrically:

if $V_i = k_i W$ $(W\times W = 0)$

$$\langle V_1, V_2\rangle_\sim = \frac{1}{(1+|W|_0^2)^{\frac{1}{3}}} \langle V_1, V_2\rangle_0$$

metric shrinks in W direction

if V_1 or $V_2 \perp W$

$$\langle V_1, V_2\rangle_\sim = (1+|W|_0^2)^{2/3} \langle V_1, V_2\rangle_0$$

expands in other 6 ~~dimensions~~ directions.

The new dual 4-form is:

$$\tilde{*}\,\tilde{\varphi} = (1+|w|_0^2)^{-\frac{1}{3}}\left[*_0\varphi_0 + *_0(w \lrcorner *_0\varphi_0) + w \lrcorner *_0(w \lrcorner \varphi_0) \right]$$

$$\tilde{\varphi} = \varphi_0 + w \lrcorner *_0 \varphi_0$$

probably elliptic in certain cases

$$d\tilde{\varphi} = 0 \iff d(w \lrcorner *_0\varphi_0) = -d\varphi$$

linear 49 eqns. 7 fncs. overdetermined

$$d\tilde{*}\tilde{\varphi} = 0 \iff \text{non-linear PDE for } w$$

Similar results hold in the $Spin_7$ case.

$$Spin_7: \quad \Phi \in \Lambda^4 = \Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{27} \oplus \Lambda^4_{35}$$

$$d\Phi \in \Lambda^5_8 \oplus \Lambda^5_{48}$$

conformal scaling eliminates this

Consider $\tilde{\Phi} = \Phi_0 + \eta_7$, similar results hold as in G_2 case.