A moment approach to analyze zeros of polynomial equations

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SYSTEM OF POLYNOMIAL EQUATIONS

$$\mathbf{S} \quad \to \quad g_1(x_1,\ldots,x_n) = 0; \ldots ; g_n(x_1,\ldots,x_n) = 0.$$

where $g_i \in \mathbf{R}[x_1,\ldots,x_n]$ for all $i = 1,\ldots,n$.

The polynomial **ideal** $I = \langle g_1,\ldots,g_n \rangle \subset \mathbf{R}[x_1,\ldots,x_n]$, generated by the family $\{g_i\}$ is assumed to be zero-dimensional, i.e. the algebraic variety

$$V_{\mathbf{C}}(I) := \{z \in \mathbf{C}^n \mid g_k(z) = 0 \quad k = 1,\ldots,n\}$$ **is finite**

Define

$$V_{\mathbf{R}}(I) := \{x \in \mathbf{R}^n \mid g_k(x) = 0 \quad k = 1,\ldots,n\}$$

be the set of **real zeros** of the system $\mathbf{S}$.

Efficient Symbolic software packages exist, especially if $\mathbf{S}$ is in *triangular form* (e.g., Aubry, Lazard, Maza, Rouillier)

Numerical solution via SDP-relaxations

Moment matrix. With $\alpha \in \mathbf{N}^n$, and $\mathbf{y}_{\alpha_1,\ldots,\alpha_n} \rightsquigarrow \int \mathbf{x}_1^{\alpha_1} \cdots \mathbf{x}_n^{\alpha_n} \, \mathbf{d}\mu$

$$M_2(y) = \left[\begin{array}{c|cc|ccc} 1 & y_{1,0} & y_{0,1} & y_{2,0} & y_{1,1} & y_{0,2} \\ \hline y_{1,0} & y_{2,0} & y_{1,1} & y_{3,0} & y_{2,1} & y_{1,2} \\ y_{0,1} & y_{1,1} & y_{0,2} & y_{2,1} & y_{1,2} & y_{0,3} \\ \hline y_{2,0} & y_{3,0} & y_{2,1} & y_{4,0} & y_{3,1} & y_{2,2} \\ y_{1,1} & y_{2,1} & y_{1,2} & y_{3,1} & y_{2,2} & y_{1,3} \\ y_{0,2} & y_{1,2} & y_{0,3} & y_{2,2} & y_{1,3} & y_{0,4} \end{array}\right]$$

In general, if $M_r(y)(i,1) = y_\alpha$ and $M_r(y)(1,j) = y_\beta$ then

$$M_r(y)(i,j) = y_{\alpha+\beta} = y_{\alpha_1+\beta_1,\ldots,\alpha_n+\beta_n}$$

Localizing matrix.

Given a polynomial $\theta : \mathbf{R}^n \to \mathbf{R}$ of degree w, with coefficient vector $\theta \in \mathbf{R}^{s(w)}$, let $M_r(\theta y)$ be the **localizing matrix**

$$M_r(\theta y)(i,j) := \sum_\alpha \theta_\alpha y_{\{\alpha(i,j)+\alpha\}}.$$

For instance, with $x \mapsto \theta(x) = 1 - x_1^2 - x_2^2$, $M_2(\theta y) =$

$$\left[\begin{array}{ccc} 1 - y_{20} - y_{02}, & y_{10} - y_{30} - y_{12}, & y_{01} - y_{21} - y_{03} \\ y_{10} - y_{30} - y_{12}, & y_{20} - y_{40} - y_{22}, & y_{11} - y_{21} - y_{12} \\ y_{01} - y_{21} - y_{03}, & y_{11} - y_{21} - y_{12}, & y_{02} - y_{22} - y_{04} \end{array}\right].$$

If $M_r(y)(i,j) = y_\beta$ then $\mathbf{M_r(\theta y)(i,j)} = \sum_\alpha \theta_\alpha \mathbf{y}_{\beta+\alpha}$ that is,

$$M_r(\theta y)(i,j) \rightsquigarrow \int x^\beta \, \theta(x) \, \mu(dx)$$

If $(1, y)$ is the vector of moments up to order $2r$ of some probability measure μ on the Borel sets of $\mathbf{R}^n$, then for every polynomial $q(x) : \mathbf{R}^n \rightarrow \mathbf{R}$ of degree at most r,

$$\langle q, M_r(y)q \rangle = \int q(x)^2 \, \mu(dx) \geq 0,$$

so that $M_r(y) \succeq 0$. Similarly,

$$\langle q, M_r(\theta y)q \rangle = \int \theta(x)q(x)^2 \, \mu(dx) \geq 0,$$

and thus $M_r(\theta y) \succeq 0$ whenever μ is supported on $\{\theta(x) \geq 0\}$.

The **$\mathbf{K}$-moment problem** identifies those vectors $\mathbf{y}$ with $M_r(y) \succeq 0$ that are **moments of a measure μ with support contained in $\mathbf{K}$**.

Dual theory in algebraic geometry = representation of polynomials, positive on a semi-algebraic set $\mathbf{K}$

With $f \in \mathbf{R}[x]$, introduce the family $\{\mathbf{Q}_i\}$ of SDP-relaxations

$$\mathbf{Q}_i \begin{cases} \min_y \sum_\alpha f_\alpha y_\alpha & \\ M_i(y) & \succeq 0 \\ M_{i-v_k}(g_k y) & = 0, \quad k = 1, \ldots, n. \end{cases}$$

and the family $\{\mathbf{Q}_i^*\}$ of their dual

$$\mathbf{Q}_i^* \begin{cases} \max_{X \succeq 0, Z_1, \ldots, Z_n} -X(1,1) - \sum_{k=1}^m g_k(0) Z_k(1,1) \\ \text{s.t.} \quad \langle X, B_\alpha \rangle + \sum_{k=1}^m \langle Z_k, C_\alpha^k \rangle = f_\alpha, \quad \forall \alpha \neq 0 \end{cases}$$

where we write

$$M_i(y) = \sum_\alpha y_\alpha B_\alpha; \quad M_{i-v_k}(g_k y) = \sum_\alpha y_\alpha C_\alpha^k, \; k = 1, \ldots m$$

I. Numerical solution via SDP-relaxations

Let $f \in \mathbf{R}[x]$ be arbitrary, fixed.

Theorem: Let I be a zero-dimensional radical ideal. Then :

(a) $\max \mathbf{Q}^*_{r_0} = \min \mathbf{Q}_{r_0} = \min \{f(x) \mid \quad x \in \mathbf{S}\}$, for some r_0, and all $r \geq r_0$.

(b) $f - f^* = q_0 + \sum_{j=1}^n q_j\, g_j$ for some s.o.s. polynomials $\{q_j\}_{j=1}^n$.

(c) Every optimal solution $y^* = \{y^*_\alpha\}$ of $\mathbf{Q}_r$ is the vector of moments of some probability measure supported on the real zeros of G.

So, when I is a radical zero-dimensional ideal, $\mathbf{Q}_r$ is exact for all $r \geq r_0$. (Lasserre (grid case), Laurent, Parrilo for the general case)

Solving Systems of polynomial equations with GLOPTIPOLY software:
CPU times in seconds and SDP-relaxation orders required to extract at least one solution

problem	n	m	d	M	N	CPU	LMI	sol
boon	6	6	4	3002	52864	1220	4	8
bifur	3	3	9	454	8717	8.20	5	2
brown	5	5	5	461	4061	6.27	3	1
butcher	7	7	4	6434	120156	-	4	mem
camera1s	6	6	2	209	952	1.33	2	2
caprasse	4	4	4	209	1285	0.58	3	2
cassou	4	4	8	4844	280151	-	8	mem
chemequ	5	5	3	461	3661	9.48	3	1
chemequs	5	5	3	124	486	6.73	2	1
cohn2	4	4	6	209	1229	0.48	3	1
cohn3	4	4	6	209	1229	0.55	3	1
comb3000	10	10	3	1000	4951	24.6	2	1
conform1	3	3	4	83	430	0.22	3	2
conform2	3	3	4	83	430	0.19	3	2
conform3	3	3	4	285	3766	3.89	5	4
conform4	3	3	4	454	8946	12.2	6	2
cpdm5	5	5	3	125	446	0.24	2	1
d1	12	12	3	-	-	-	3	dim
des18_3	8	8	3	12869	303945	-	4	mem
des22_24	10	10	2	1000	5016	77.2	1	1

problem	n	m	d	M	N	CPU	LMI	sol
discret3	8	8	2	44	89	0.31	1	1
eco5	5	5	3	461	3661	5.98	3	1
eco6	6	6	3	923	7980	57.4	3	1
eco7	7	7	3	1715	15921	256	3	1
eco8	8	8	3	3002	29565	1310	3	1
fourbar	4	4	4	69	229	0.16	2	1
geneig	6	6	3	923	7602	33.2	3	1
heart	8	8	4	3002	31545	1532	3	2
i1	10	10	3	1000	4366	44.1	2	1
ipp	8	8	2	494	2385	6.42	2	1
katsura5	6	6	2	209	952	0.74	2	1
kinema	9	9	2	714	3520	26.4	2	1
kin1	12	12	3	-	-	-	3	dim
ku10	10	10	2	1000	5016	72.5	2	1
lorentz	4	4	2	209	1705	0.64	2	2
manocha	2	2	8	90	826	1.27	6	1
noon3	3	3	3	83	430	0.22	3	1
noon4	4	4	3	209	1285	0.65	3	1
noon5	5	5	3	461	3241	4.48	3	1

problem	n	m	d	M	N	CPU	LMI	sol
proddeco	4	4	4	69	229	0.11	2	1
puma	8	8	2	3002	35505	1136	3	4
quadfor2	4	4	4	209	1495	0.75	3	2
quadgrid	5	5	5	461	3641	10.52	3	1
rabmo	9	9	5	5004	51703	-	3	mem
rbpl	6	6	3	923	7602	36.9	3	1
redeco5	5	5	2	20	41	0.16	1	1
redeco6	6	6	2	27	55	0.13	1	1
redeco7	7	7	2	35	71	0.14	1	1
redeco8	8	8	2	44	89	0.13	1	1
rediff3	3	3	2	9	19	0.09	1	1
reimer5	5	5	6	6187	264516	-	6	mem
rose	3	3	9	679	16681	79.5	7	2
s9_1	8	8	2	494	2385	5.45	2	1
sendra	2	2	7	65	453	0.34	5	1
solotarev	4	4	3	69	257	0.24	2	1
stewart1	9	9	2	714	3520	20.4	2	2
stewart2	12	10	2	1819	9191	372	2	1
trinks	6	6	3	209	925	0.78	2	1
virasoro	8	8	2	44	89	0.16	1	1

II. Characterization of zeros

Problems : Give conditions on the coefficients of $\{g_i\}$ to ensure that

- $V_{\mathbf{C}}(I) \equiv V_{\mathbf{R}}(I)$, i.e., $\mathbf{S}$ has only real zeros

- $V_{\mathbf{C}}(I) \equiv V_{\mathbf{R}}(I)$ and $V_{\mathbf{R}}(I) \subseteq \mathbf{K}$ for some specified semi-algebraic set $\mathbf{K} \subset \mathbf{R}^n$.

- $V_{\mathbf{C}}(I) \subseteq \mathbf{K}$ for some specified subset $\mathbf{K} \subset \mathbf{C}^n$ (or semialgebraic set $\mathbf{K}'$ of $\mathbf{R}^{2n}$).

The univariate case.

Let $g \in \mathbf{R}[x]$ with $x \mapsto g(x) = x^{n+1} + a_n x^n + \ldots + a_0$.

Conditions on the $\{a_i\}$ for the n zeros $\{x(j)\} \subset \mathbf{C}$ (counting multiplicities) of g to be all real and contained in the interval $[u, v] \subset \mathbf{R}$.

Define the Newton sums (counting multiplicities)

$$s_k := \frac{1}{n} \sum_{j=1}^{n} x(j)^k \qquad k = 0, 1, \ldots,$$

The s_k's are the **moments** of the **probability measure**

$$\mu := \frac{1}{n} \sum_{j=1}^{n} \delta_{x(j)} \quad \text{counting multiplicities}$$

Hence, write that μ is supported on $\mathbf{K} := [u, v]$!!

Let $H(n,s)$ and $M(n,s)$ be the respective Hankel matrices

$$\left[\begin{array}{cccc} 1 & s_1 & \cdots & s_n \\ s_1 & s_2 & \cdots & s_{n+1} \\ & & & \\ s_n & s_{n+1} & \cdots & s_{2n} \end{array}\right]; \quad \text{and} \left[\begin{array}{cccc} s_1 & s_2 & \cdots & s_{n+1} \\ s_2 & s_2 & \cdots & s_{n+2} \\ & & & \\ s_{n+1} & s_{n+2} & \cdots & s_{2n+1} \end{array}\right].$$

Theorem : [Lasserre (J. Alg. Comb. (2002))]

(i) All the zeros of g are real **iff** $M(n,s) \succeq 0$.
(and rank($M_n(s,y)$)) are distinct.)

(ii) All the zeros of g are real and contained in $[u,v]$ **iff :**

$$v.H(n,s) \succeq M(n,s) \succeq u.H(n,s)$$

The (univariate) complex case

The complex moment matrix : Consider the basis of monomials

$$1, z, \overline{z}, z^2, z\overline{z}, \overline{z}^2, \ldots, z^n, z^{n-1}\overline{z}, \ldots z\overline{z}^{n-1}, \overline{z}^n, \ldots$$

of the complex polynomials $q \in \mathbf{C}[z, \overline{z}]$, that is,

$$z \mapsto q(z, \overline{z}) = \sum q_{ij} \overline{z}^i z^j \quad \text{for finitely many } q_{ij}.$$

For a measure μ on $\mathbf{C}$ let $y_{ij} := \int \overline{z}^i z^j \, d\mu$ for all $i, j = 0, 1, \ldots$, and let $M_r(y)$ be its moment matrix, i.e.,

$$[M_r(i,1) = y_{pq} \text{ and } M_r(1,j) = y_{vw}] \Rightarrow \quad M_r(i,j) = y_{p+v,q+w}.$$

Then, $\forall f \in \mathbf{C}[z, \overline{z}]$ with degree $\leq r$, and coefficient vector f,

$$\langle \mathbf{f}, \mathbf{M_r f} \rangle = \int \overline{f}(\mathbf{z})\, \mathbf{f}(\mathbf{z})\, \mu(\mathbf{dz}) = \int |\mathbf{f}|^2 \, \mathbf{d}\mu,$$

so that $M_r(y)$ is Hermitian and positive semidefinite ($M_r(y) \succeq 0$)

Let $z \mapsto p(z) := a_0 + a_1 z + \ldots a_n z^n + z^{n+1} \in \mathbf{R}[z]$ be a polynomial with real coefficients, and let

$$\mathbf{K} := \{z \in \mathbf{C} \mid \quad g_k(z, \overline{z}) \geq 0; \quad k = 1, \ldots, m\}$$

be a given (nonnecessarily compact) semialgebraic set of $\mathbf{C}$.

Problem: Under which condition on the coefficients $\{a_j\}$ do we have all the zeros of p contained in $\mathbf{K}$?

Example : $\mathbf{K} = \{z \in \mathbf{C} \mid \quad z + \overline{z} \leq 0\}$ for stability of linear systems, which gave the Routh-Hurwitz and Liénard criteria involving determinants, linear in the coefficients of p.

Some necessary and sufficient conditions provided in the pioneering work of Kalman, later extended by Mazko, Gutman and Jury, Chilali and Gahinet, ...

Let $\{z(k)\}_{k=1}^{n} \subset \mathbf{C}$ be the n zeros of p (counting their multiplicity), and define the probability measure μ on $\mathbf{C}$

$$\mu := \frac{1}{n} \sum_{k=1}^{n} \delta_{z(k)}; \quad y_{ij}^{*} = \int \overline{z}^{i} z^{j} \, d\mu = y_{ji}^{*}.$$

Hence, $M_r(y^*) \succeq 0$ is a real symmetric matrix.

$$y_{0j}^{*} = y_{j0}^{*} := \int z^{j} \, d\mu = s_j \quad j_{th} \text{ Newton sum } = f_j(a)$$

The Newton sums s_j's are known and (easy to compute) functions of the a_j's, but **not** the y_{ij}^{*}.

However, because $p(z) = 0$, we have

$$z^p = \sum_{k=0}^{n} \beta_k(p) z^k; \qquad \overline{z}^p = \sum_{k=0}^{n} \beta_k(p) \overline{z}^k$$

for some $\{\beta_k(p)\} \subset \mathbf{R}$.

Hence, for all p, q we have

$$(*) \quad y^*_{pq} = \sum_{0 \le i,j \le n} \gamma_{ij}(p,q)\, y^*_{ij} \qquad \text{for some } \{\gamma_{ij}(p,q)\} \subset \mathbf{R}.$$

and the $\gamma_{ij}(p,q)$'s are **easy** to compute.

So define the moment matrix $M_r(y)$ (or $M_r(y,s)$)

$$\text{Ex}: \; M_2(y) = M_2(y,s) = \begin{bmatrix} 1 & s_1 & s_1 & s_2 & y_{11} & s_2 \\ s_1 & y_{11} & s_2 & y_{12} & y_{12} & s_3 \\ s_1 & s_2 & y_{11} & s_3 & y_{12} & y_{12} \\ s_2 & y_{12} & s_3 & y_{22} & y_{13} & s_4 \\ y_{11} & y_{12} & y_{12} & y_{13} & y_{22} & y_{13} \\ s_2 & s_3 & y_{12} & s_4 & y_{13} & y_{22} \end{bmatrix}$$

with $\mathbf{y}$ **unknown** in lieu of $\mathbf{y}^*$, and using (*) to have

$$y_{pq} = \sum_{0 \leq i,j \leq n} \gamma_{ij}(p,q)\, y_{ij}$$

Therefore, $M_r(y,s)$ contains only the $n(n+1)/2$ variables $\{y_{ij}\}$, with $1 \leq i \leq\leq n$.

For $z \mapsto g_k(z,\overline{z}) = \sum_{u,v} g_k(u,v)\overline{z}^u z^v$, define the localizing matrices $\{M_r(g_k,y,s)\}$

$$M_r(y,s)(i,j) = y_{pq} \Rightarrow \quad M_r(g_k,y,s)(i,j) = \sum g_k(u,v)\, y_{p+u,q+v}$$

for all $k = 1,\ldots,m$, and again using (*).

Theorem : Let μ be the uniform probability measure on the zeros of p (counting multiplicities). Then:

(a) The SDP constraint $M_{2n}(y,s) \succeq 0$ yields a unique solution y^*, the vector of moments of μ (up to order $2n$).

(b) All the zeros of p are contained in $\mathbf{K}$ if and only if $M_{2n}(g_k,y^*,s) \succeq 0$, for all $k = 1,\ldots,m$

The proof uses a nice result of Curto and Fialkow on flat positive extensions of moment matrices.

Multivariable case $\mathbf{C}^n$

Consider the system $\mathbf{S}$ of polynomial equations

$$h_1(x_1) = 0;\ h_2(x_1, x_2) = 0; \ldots;\ h_n(x_1, x_2, \ldots, x_n) = 0$$

in **triangular form**, where :

$$h_k(x) = h_{k1}(x_1, \ldots, x_{k-1})\, x_k^{r_k} + h_{k2}(x_1, \ldots, x_k); \quad k = 2, \ldots, n$$

and $h_{k1}(x_1, \ldots, x_{k-1}) \neq 0$ whenever $h_i(x) = 0, k = 1, \ldots k-1$.

(i) Every system of polynomial equations associated with a zero-dimensional ideal is a finite union of such triangular systems

(ii) Symbolic computation packages can obtain this form.

Let $\{z(k)\}_{k=1}^{t} \subset \mathbf{C}^n$ be the t complex zeros of the system $\mathbf{S}$. Then, if one defines the *generalized* Newton sums

$$s_\alpha := \int z_1^{\alpha_1} \cdots z_n^{\alpha_n}\, d\mu, \quad \text{with } \mu := \frac{1}{t}\sum_{k=1}^{t} \delta_{z(k)}$$

(**) One may compute $\{s_\alpha\}$ recursively as rational fractions of the coefficients of the polynomials $\{h_k\}$ that define $\mathbf{S}$.

Similarly, let

$$y^*_{\alpha\beta} = \int \overline{z}^{\alpha}\, z^{\beta}\, \mu(dz) \qquad \alpha, \beta \in \mathbf{N}^n.$$

As in the one-dimensional case,

$$(*) \qquad y^*_{\eta\delta} = \sum_{\alpha_i, \beta_i < r_i\, \forall i} \gamma_{\alpha\beta}(\eta, \delta)\, y^*_{\alpha\beta}$$

So, once the $\{y^*_{\alpha\beta}\}$ (with $\alpha_j, \beta_j < r_j$ for all $j = 1, \ldots, n$) are known, then all the $y^*_{\alpha\beta}$'s are known via (*)!

Let $\mathbf{K} := \{z \in \mathbf{C}^n \mid \quad g_k(z, \overline{z}) \geq 0,\ k = 1, \ldots, m\} \subset \mathbf{C}^n$ be given.

One defines the multivariable analogues of the moment matrix $M_r(y, s)$ and localizing matrices $M_{g_k,r}(y, s)$, with $\mathbf{y}$ unknown in lieu of $\mathbf{y}^*$ and using (*)

$$(*) \qquad\qquad y_{\eta\delta} = \sum_{\alpha_i, \beta_i < r_i\ \forall i} \gamma_{\alpha\beta}(\eta, \delta)\, y_{\alpha\beta}$$

so that $M_r(y, s)$ and $M_{g_k,r}(y, s)$ contain only the variables $\{y_{\alpha\beta}\}$ with $\alpha_i, \beta_i \leq r_i$ for all $i = 1, \ldots, n$.

Theorem : Let μ be the uniform probability measure on the zeros of $\mathbf{S}$ (counting multiplicities). Let $r_0 := \sum_{j=1}^{n} r_j - 1$. Then:

(a) The SDP constraint $M_{2r_0}(y, s) \succeq 0$ yields a unique solution y^*, the vector of moments of μ (up to order $2r_0$).

(b) rank($M_{2r_0}(y, s)$ gives the number of distinct zeros of $\mathbf{S}$.

(c) Let $I = \langle g_1, \ldots, g_n \rangle$. $f \in \mathbf{R}[z]$ of degree $2p$ or $2p - 1$. Then

$$f \in \sqrt{I} \Leftrightarrow M_p(\mu^*) f = 0.$$

(d) All the zeros of p are contained in $\mathbf{K}$ if and only if $M_{2r_0}(g_k, y^*, s) \succeq 0$, for all $k = 1, \ldots, m$

Again, we use Curto and Fialkow's result on flat positive extensions of moment matrices.

Real zeros

Let $\mathbf{K} \subset \mathbf{R}^n$ be the semi-algebraic set

$$\mathbf{K} = \{x \in \mathbf{R}^n \,|\quad g_j(x) \geq 0, \quad j = 1, \ldots, m\}.$$

Let $M_r(y)$ be the real moment matrix with rows and columns indexed in the basis $1, x_1, \ldots, x_n, x_1^2, \ldots, x_n^r$.

If μ is a probability measure with support on the real zeros of $\mathbf{S}$ and $y = \{y_\alpha\}$ is the vector of its moments, one has

$$(*) \qquad\qquad y_\beta = \sum_{\alpha_i < r_i \,\forall i} \gamma_\alpha(\beta)\, y_\alpha$$

for some real coefficients $\{\gamma_\alpha(\beta)\}_\alpha$, easy to compute.

Replace y_β with (*) in $M_r(y)$, whenever $\beta_j > r_j$, for some j. So, $M_r(y)$ has only finitely many unknowns $\{y_\alpha\}$, for all r.

Theorem : Let $r_0 := \sum_{j=1}^{n} r_j - 1$. Then:

(a) The number s_0 of distinct zeros of $\mathbf{S}$ is the maximum rank of the real moment matrices $M_{r_0}(y)$ which are positive semidefinite.

(b) Let $M_{r_0}(y^*) \succeq 0$ with rank$(M_{r_0}(y^*)) = s_0$. Then, $M_r(y^*) \succeq 0$ and $M_r(y^*)$ has rank s_0 for all $r \geq r_0$.

(c) Let $I = \langle g_1, \ldots, g_n \rangle$, and $f \in \mathbf{R}[x]$ be of degree $\leq r$. Then

$$f \in I(V_{\mathbf{R}}(I)) \Leftrightarrow M_r(y^*)f = 0.$$

(d) All the zeros of p are contained in $\mathbf{K}$ if and only if $M_{r_0}(g_k, y^*) \succeq 0$, for all $k = 1, \ldots, m$

(**) Solving $M_{r_0}(y) \succeq 0$ with $M_{r_0}(y)$ of maximal rank is NP-hard!!

Note in passing that

$\mathbf{S}$ has only real zeros if and only if $M_{r_0}(s) \succeq 0$, where $s = \{s_\alpha\}$ is the (known) vector of Newton's sums of $\mathbf{S}$.