MSRI
Mathematical Sciences Research Institute

17 Gauss Way Berkeley, CA 94720-5070 p: 510.642.0143 f: 510.642.8609 www.msri.org

NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: A. Seceleanu Email/Phone: aseceleanu2@math.unl.edu

Speaker's Name: Claudia Miller

Talk Title: Duality for Koszul Homology over Gorenstein Rings

Date: 08/22/12 Time: 10:45 am / pm (circle one)

List 6-12 key words for the talk: Gorenstein, Koszul, homology, duality, spectral sequence

Please summarize the lecture in 5 or fewer sentances: When a ring is Gorenstein, certain modules exhibit nice duality properties. The lecture explains various versions of duality for Koszul homology.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
    - Computer Presentations: Obtain a copy of their presentation
    - Overhead: Obtain a copy or use the originals and scan them
    - Blackboard: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
    - Handouts: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Claudia Miller - Duality for Koszul Homology

R = Gorenstein ring
I = ideal with minimal generators $x_1, \dots, x_e = \underline{x}$, grade $I = g$

Koszul homology $H_i(\underline{I}) = H_i(K(\underline{x}))$.
$\uparrow$ Koszul complex

$H.(I) = \bigoplus_{i=0}^{e-g} H_i(I)$ graded algebra

Multiplication maps $H_i \times H_{e-g-i} \longrightarrow H_{e-g}$, $\forall i$

$\Leftrightarrow H_i \xrightarrow{\phi_i} \mathrm{Hom}(H_{e-g-i}, H_{e-g}) \cong \mathrm{Ext}^g(H_{e-g-i}, R)$

Want to know when this is a perfect pairing. H_{e-g-i} $\overset{\text{not}}{\neq}$

We say $H.$ satisfies Poincaré duality (PD) if we have perfect pairings $\forall i$.

$\boxed{\text{For } I = m:}$

Thm (classic)
R regular $\Leftrightarrow H_1 = 0$ $(\Leftrightarrow (x_1, \dots, x_e) = $ reg. seq.$)$

Thm (Tate '57, Asmus '59)
R is ci $\Leftrightarrow H. = \Lambda H_1$

Thm (Avramov - Golod '71)
R is Gorenstein $\Leftrightarrow H.$ satisfies PD

$\boxed{\text{For general } I}$:

Thm (Herzog) If R is Gorenstein and H_i is CM $\forall i$, then $H.$ satisfies PD.
actually just need
H_j to be CM $\forall j \le i$ - fixed
then $H.$ satisfies PD up to i

Examples: • licci ideals (Huneke)
• pd ≤ 2 (Avramov - Herzog)

• $\ell-g$ small (Nagel - Puthenpurakal)

<u>Thm</u> (Huneke)
If H_i satisfies Serre's S_2 condition (as an R/I-module $\forall i$)
then get P.D.

In general, ϕ_i fails to be an isomorphism, but it's not far:

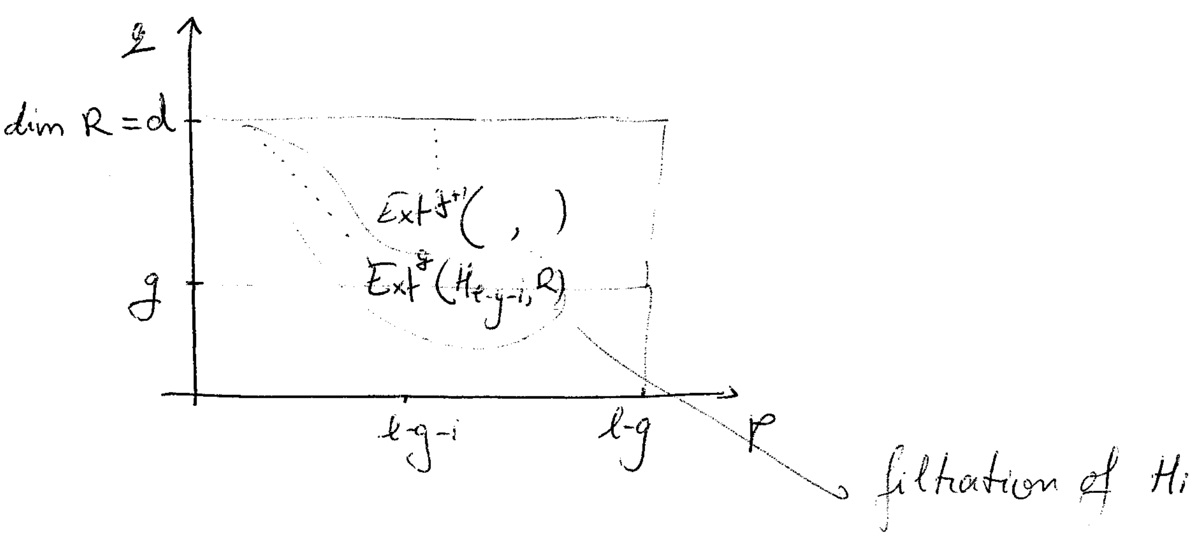
<u>Thm</u> (Chardin - graded case, Miller - Rahmati - Striuli)
Any I, $\exists$ isom:

$$H_i^{\vee} \xleftarrow{\cong} H_{\ell-g-i}^{\vee\vee}$$

<u>Cor</u> Fix i. If $H_i(I)$ satisfies S_2, $H_i \xrightarrow{\cong} H_{\ell-g-i}^{\vee}$

Proof via a classic spectral sequence:

$R \xrightarrow{\simeq} J^*$ injective resolution
bicomplex $\mathrm{Hom}(K_\bullet, J^\bullet)$ yields $E_2^{pq} = \mathrm{Ext}^q(H_p, R) \Rightarrow H_{\ell-(p+q)}$



1) All above row g has dim $<$ dim R/I, so $\mathrm{Ext}^g(-, R)$ kills it
2) $\mathrm{Ext}^q(H_p, R) = H_m^{d-q}(H_p)$; vanishing measures depth of H_p
local cohomology

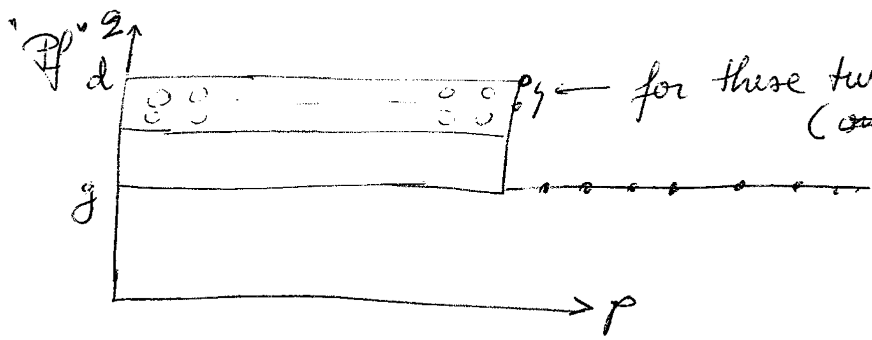
(S_2 is sufficient)

<u>Conversely</u>: if you know $H.$ satisfies duality (edge map iso)

1) $E^{\infty}_{p,q} = 0$ $\forall (p,q)$ on the $p+q=i$ diagonal for $q > g$

2) all maps from bottom row are 0

<u>Thm</u> If $\forall i$ H_i satisfies S_2 (as R/I-module) and $S_{\frac{\dim R/I}{2}}$ (i.e. $S_{\frac{d-g}{2}}$), then H_i is CM $\forall i$.



← for these two positions $E_2 = E^{\infty}$ (~~only row~~ + all maps in and out are zero, including from bottom row)

use <u>Hartshorne – Ogus</u>:

If M is an $S = R/I$-module

depth M_P + depth $M_P^{\vee} \geq \dim S_P + 2$ $\forall P$ ($ht \geq 2$)

then M is CM.

In the two highlighted positions $E^2_{pq} = E^{\infty}_{pq} \overset{\text{duality}}{=} 0$

This gives 2 more "bits of depth".

Then use induction col. by column.

<u>Thm</u> (Extends a sliding depth result of Herzog – Vasconcelos – Villarreal) Fix $j \geq 0$.

If (a) $\mu(I_P) \leq ht\, P + j$ $\forall P$

(b) H_i satisfy sliding depth, depth $H_i \geq d - \ell + i + h$

$h = \lceil \frac{j+1}{2} \rceil$

(c) H_i satisfies S_2

Then H_i CM $\forall i$.