Fine estimates in Dvoretsky's thm (Gideon Schechtman)

Dvoretsky's thm (60, version by Milman '71): (1)
$\forall \varepsilon > 0$, $\exists\ c(\varepsilon) > 0$ s.t. if $\boxed{k \le c(\varepsilon) \log n}$ then
$\forall$ n-dimensional symmetric convex body K in $\mathbb{R}^n$, $\exists$
a k-dimensional subspace L s.t. $K \cap L$ is ε-Euclidean,
ie. for some $R > 0$,
$$R(B_2^n \cap L) \subseteq K \cap L \subseteq (1+\varepsilon) R (B_2^n \cap L).$$

Eg: if $K = B_1^n$ and we want to put balls inside and outside, then the ratio of their radii is $\sqrt{n}$. But if we go to a subspace, the ratio comes down to $1+\varepsilon$.

Equivalently: if $k \le c(\varepsilon) \log n$ then $\ell_2^k \overset{1+\varepsilon}{\hookrightarrow} X$ for every n-dimensional normed space X.

(where $U \overset{K}{\hookrightarrow} V$ means $\exists$ linear $T : U \to V$ s.t. $\frac{1}{K}\|x\| \le \|Tx\| \le \|x\|$).

(To see that these are equivalent uses the fact that every k-dim ellipsoid has a $k/2$-dim section that is a Euclidean ball).

Actually, Milman proved something stronger:
$\forall \varepsilon > 0$ $\exists\ c(\varepsilon)$ s.t. $\forall$ n-dim normed spaces $X = (\mathbb{R}^n, \|\cdot\|)$ with $B_2^n \subseteq B_{\|\cdot\|}$, if we set $E = E_{\|\cdot\|} = \mathbb{E}\left\|\sum_{i=1}^n g_i e_i\right\|$
where $g_1, \ldots, g_n$ are iid $N(0,1)$ and $\boxed{k \le c(\varepsilon) E^2}$ then $\ell_2^k \overset{1+\varepsilon}{\hookrightarrow} X$. (2)

It turns out that $E \ge \sqrt{\log n}$, so we recover Dvoretsky's thm.

$\to$ (3)

Examples:

For $X = \ell_p^n$, if $\boxed{k \le c(\varepsilon) \begin{cases} n & 1 \le p < 2 \\ p n^{2/p} & 2 < p \end{cases}}$

then $\ell_2^k \xhookrightarrow{1+\varepsilon} \ell_p^n$.

Questions:

- What is the behavior of $c(\varepsilon)$ in (1)
- " " (2)
- " " (3)
- What if you want most sections of dim K to be ε-Euclidean

The proof of Milman gives $c(\varepsilon) \ge c\varepsilon^2/\log 1/\varepsilon$
improved to $c\varepsilon^2$ by Gordon

Figiel: $\forall\ \varepsilon > 0$ and n large enough $(n > \varepsilon^{-4})$,
$\exists$ n-dim X s.t. $X \overset{2}{\sim} \ell_2^n$ and such that
if $V \subseteq X$, $\dim V = k$, $V \overset{1+\varepsilon}{\sim} \ell_2^k$ then $k \le \varepsilon^2 n$.

$\|\cdot\|_2 + \|\cdot\|_p$ for $p \ne 2$.

(Remark: $X \overset{2}{\sim} \ell_2^n \Rightarrow E \sim \sqrt{n}$).
This basically solves the second question above.

For the first question, it is known that $c(\varepsilon) \le \frac{c}{\log 1/\varepsilon}$ for $\|\cdot\|_\infty$. (And $k \sim \log n/\log 1/\varepsilon$ is the right estimate for $\|\cdot\|_\infty$).

Claim: In (1), $c(\varepsilon) \ge \frac{c\,\varepsilon}{(\log 1/\varepsilon)^2}$.

In the case of ℓ_p:

- for $p=4$, $c(\varepsilon) = \text{const}$ (König)
- for p even, $c(\varepsilon) \sim \varepsilon^{4/p}$

For the probabilistic version, can't do better than $c(\varepsilon) \sim \varepsilon$.

A little bit about the proof of the claim:
If $\varepsilon^2 E^2 \geq \varepsilon / (\log 1/\varepsilon)^2 \log n$ then we are done by Milman's thm. So assume the opposite.

$$E = \mathbb{E} \left\| \sum g_i e_i \right\| \leq L \sqrt{\log n} \quad \text{with } L = \frac{1}{\sqrt{\varepsilon} \log 1/\varepsilon}$$

Main observation: if $x_1, \dots, x_n \in X$, $\|x_i\| = 1$ and $\mathbb{E} \| \sum g_i x_i \| \leq L \sqrt{\log n}$ then $\ell_\infty^{n^{1/4}} \xhookrightarrow{100L} X$.
This is similar to a thm of Alon-Milman. A theorem of James lets us reduce the constant $100L$ to get

$$\ell_\infty^{n^{c\varepsilon}} \xhookrightarrow{1+\varepsilon} X.$$

Then note that $\ell_2^k \hookrightarrow \ell_\infty^{n^{c\varepsilon}}$ (with a good dependence on ε). $\square$.