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Name: KAROL KOZIOL Email/Phone: (315) 569-6968

Speaker's Name: ILA VARMA

Talk Title: LOCAL-GLOBAL COMPATIBILITY OF REGULAR ALGEBRAIC CUSPIDAL AUTOMORPHIC REPRESENTATIONS WHEN $\ell \neq p$

Date: 8 / 14 / 14 Time: 3 : 30 am / pm (circle one)

List 6-12 key words for the talk: AUTOMORPHIC REPRESENTATIONS, LOCAL-GLOBAL COMPATIBILITY, GLOBAL LANGLANDS CORRESPONDENCE

Please summarize the lecture in 5 or fewer sentences: STARTING WITH A REGULAR, ALGEBRAIC, CUSPIDAL AUTOMORPHIC REPRESENTATION OVER A ~~REAL~~ TOTALLY REAL OR CM FIELD, A CONSTRUCTION OF HARRIS-LAN-TAYLOR-THORNE AND SCHOLZE CONSTRUCTS ~~AN~~ THE ASSOCIATED GLOBAL GALOIS REPRESENTATION. IN THIS TALK, IT IS SHOWN THAT THIS ASSOCIATION IS COMPATIBLE WITH THE LOCAL LANGLANDS CORRESPONDENCE AT ALL PRIMES NOT ABOVE P.

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Local-global compatibility for ramified primes

Ila Varma

August 18, 2014

Local-global compatibility characterizes the Global Langlands correspondence in terms of the Local Langlands correspondence at each prime. Today I will discuss this compatibility statement in general as well as in the new instances of Global Langlands proved by Harris-Lan-Taylor-Thorne and Scholze.

Local Langlands correspondence for GL_n

Let F be a number field, p a rational prime, v a place of F over the rational prime ℓ s.t. $v \nmid p$. For the sake of this lecture, assume v splits over F^+. The Weil group will be denoted $W_{F_v} \subset \mathrm{Gal}(\overline{F}_v/F_v)$. We will denote Weil-Deligne representations of W_{F_v} as triples (r, V, N) where $r : W_{F_v} \to \mathrm{GL}(V)$ and N is the monodromy operator.

We say (r, V, N) is **Frobenius semisimple** iff r is semisimple, and we say (r, V, N) is **semisimple** iff r is semisimple and $N = 0$.

It is a theorem of Grothendieck that continuous p-adic Galois representations of W_{F_v} can be encoded as Weil-Deligne representations when $v \nmid p$, and this is the language used in stating the Local Langlands correspondence. We can now state the bijection of the Local Langlands correspondence.

Theorem 1 (Harris-Taylor, Henniart)**.** *There is a bijection between*

$$\{\textit{irreducible smooth rep's of } \mathrm{GL}_n(F_v) \textit{ over } \mathbb{C}\} \longleftrightarrow \{\textit{Frob-s.s. WD repns of } W_{F_v} \textit{ over } \mathbb{C}\}$$

For example, principal series associated to two characters of $F_v^\times$ whose quotient is not equal to the norm or its inverse has Weil-Deligne representation equal to the direct sum of these characters with $N = 0$.

Global Langlands conjecture

We will use LLC in our description of the Global Langlands correspondence. Fix $\imath : \mathbb{C} \cong \overline{\mathbb{Q}}_p$

Conjecture 1 (Langlands, Fontaine-Mazur)**.** *There should be a bijection*

$$\{\textit{algebraic cuspidal aut. rep'ns of } \mathrm{GL}_n(\mathbb{A}_F)\} \longleftrightarrow \left\{\begin{matrix} \textit{irred. cts. Galois rep'ns } G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p) \\ \textit{unram. at almost all places and de Rham at } p \end{matrix}\right\}$$

Note that cuspidal corresponds to irreducible. Algebraic is a condition at the infinite places, namely that the infinitesimal character has integral Harish-Chandra parameters. This corresponds to the de Rham at p condition, which can be thought of as the condition that the representation comes from geometry (at least conjecturally).

The above bijection should satisfy the following being equivalent:

1. Starting with $\pi = \bigotimes_v' \pi_v$, restrict to π_v and apply local Langlands after a specified normalization, namely

$$\mathrm{rec}(\pi_v \otimes |\det|_v^{(1-n)/2})$$

2. Starting with π, take the corresponding (global) Galois representation $R_p(\pi) : G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$, restrict to G_{F_v} and apply Grothendieck's WD.

In other words, we want $\pi \mapsto R_p(\pi)$ to satisfy

$$\mathrm{WD}(R_p(\pi)|_{G_{F_v}})^{F-ss} \cong \imath^{-1}\,\mathrm{rec}(\pi_v \otimes |\det|^{(1-n)/2}). \qquad (*)$$

The above statement is what's referred to as Local-Global compatibility.

1 Recent new instances of Global Langlands

Theorem 2 (Harris-Lan-Taylor-Thorne, Scholze)**.** *Assume F is CM or totally real, and assume that π is a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. (Note that regular algebraic is the added condition that the Harish-Chandra parameters are distinct). Then there exists $R_p(\pi) : G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ such that at places of F not above p for which both F and π are unramified,*

$$\mathrm{WD}(R_p(\pi)|_{G_{F_v}})^{F-ss} \cong \imath^{-1}\,\mathrm{rec}(\pi_v \otimes |\det|^{(1-n)/2}). \qquad (*)$$

In this case, note that the statement of local-global compatibility is just about the image of Frobenius, and $N = 0$. Note that previous to these results, one needed to further assume conjugate self-duality, namely that $\pi = \pi^{c,\vee}$.

The strengthened local-global compatibility result is:

Theorem 3 (V)**.** *For all primes $\nmid p$,*

$$\mathrm{WD}(R_p(\pi)|_{G_{F_v}})^{ss} \cong \imath^{-1}\,\mathrm{rec}(\pi_v \otimes |\det|^{(1-n)/2}). \qquad (*)$$

Remark. 1. The statement above ignores N completely. In fact, we can "bound" N on the Galois side by N on the automorphic side.

2. We can further prove that $R_p(\pi)$ is de Rham.

Argument

We will now p-adically interpolate Galois representations associated to those π' satisfying $\pi' = (\pi')^{c,\vee}$ to get Galois representations for general RAC π that are not necessarily conjugate self-dual. Start with such a π on GL_n.

Denote by $G = GU(n,n)$ the quasisplit unitary similitude group associated to F^{2n} and alternating form $\begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$. It has parabolic $P = \mathrm{GL}_1 \times \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ with Levi $L = \mathrm{GL}_1 \times \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$. However, note that $L \cong \mathrm{GL}_1 \times \mathrm{RS}^F_{\mathbb{Q}} \mathrm{GL}_n$.

For all M sufficiently large positive integer, let

$$\Pi(M) = \mathrm{Ind}_{P(\mathbb{A}^{p,\infty})}^{GU(n,n)(\mathbb{A}^{p,\infty})}(1 \times (\pi \otimes ||\det||^M)^{\infty}).$$

Its base-change to GL_{2n} will look like $(\pi \otimes ||\det||^M) \oplus (\pi \otimes ||\det||^M)^{c,\vee}$ away from infinity (and p). HLTT prove that $\Pi(M)$ is a subquotient of a space of overconvergent p-adic automorphic forms of $GU(n,n)$. We now introduce such a space.

2 Some compactifications

Let X_U be the Shimura variety associated to $GU(n,n)$ and some neat open compact U. For the sake of this talk, assume it is prime-to-p level structure.

Now let $\mathcal{X}_U^{\min}$ be an integral model of the minimal compactification of X_U. It is a normal projective flat scheme defined over $\mathbb{Z}_{(p)}$. Thus, there is an ample line bundle, ω_U whose pullback to X_U is identified with the determinant of the Hodge bundle. Over $\mathbb{F}_p$, it has a canonical global section Hasse_U such that

$$g_* \,\mathrm{Hasse} = \mathrm{Hasse} \quad \forall g \in GU(n,n)(\mathbb{A}^{p,\infty})$$

Kai-Wen Lan has constructed a normal quasi-projective scheme over $\mathbb{Z}_{(p)}$ which is a partial minimal compactification of the ordinary locus X_U, denoted $\mathcal{X}_U^{\mathrm{ord,min}}$. Over $\mathbb{F}_p$, this coincides with the nonzero locus of Hasse.

For any algebraic representation ρ of L of highest weight μ, let $\mathcal{E}_\mu^{\mathrm{sub}} = \mathcal{E}_\rho^{\mathrm{sub}}$ denote the subcanonical extension of the automorphic vector bundle associated to ρ on $\mathcal{X}_U^{\mathrm{min}}$. It is constructed as the pushforward of the canonical extension $\mathcal{E}_\rho^{\mathrm{can}} \otimes \mathcal{I}_{\partial X_{\Delta,U}}$ from a toroidal compactification of X_U that maps to $\mathcal{X}_U^{\mathrm{min}}$.

The space of p-adic automorphic forms of weight μ is then $H^0(\mathcal{X}^{\mathrm{min,ord}}, \mathcal{E}_\mu^{\mathrm{sub}})$.

Proposition 4 (HLTT)**.** *$\Pi(M) \in H^0(\mathcal{X}^{\mathrm{min,ord}}, \mathcal{E}_{\mu_0}^{\mathrm{sub}})$ for some non-classical μ_0 and has finite slope.*

The analogue in GL_2 of this space is $H^0(X_1(N)^{\mathrm{ord}}, \omega^{\otimes k}(-\mathrm{cusps}))$, where k is negative. In particular, we don't expect to find any classical automorphic forms of weight μ_0, i.e. we expect

$$H^0(\mathcal{X}^{\mathrm{min}}, \mathcal{E}_{\mu_0}^{\mathrm{sub}}) = 0.$$

If μ is satisfies a "classicality condition" (analogous to $k \geq 2$), then $H^0(\mathcal{X}^{\mathrm{min}}, \mathcal{E}_\mu^{\mathrm{sub}})$ consists of classical cuspidal automorphic representations Π' whose base change to GL_{2n} satisfies $\mathrm{BC}(\Pi') = \mathrm{BC}(\Pi')^{c,\vee}$, i.e. are conjugate self-dual. With this added condition,

Proposition 5 (Chenevier-Harris,Shin,BLGHT,CHT,TY,Caraiani)**.** *Associated to each Π',* *there exists a Galois representation,*

$$R_p(\Pi') : G_F \to \mathrm{GL}_{2n}(\mathbb{Q}_p)$$

satisfying full local-global compatibility, i.e.

$$\mathrm{WD}(r_p(\Pi')|_{G_{F_v}})^{F-ss} \cong \mathrm{rec}(\mathrm{BC}(\Pi'_v) \otimes |\det|^{(1-2n)/2}).$$

Furthermore, there is a representation $S := \wedge^{n[F:\mathbb{Q}]} \mathrm{Std}^\vee$ whose highest weight is $w = (0, (-1, \ldots, -1))$, and we have the relation that

$$\mathcal{E}^{\mathrm{sub}}_{\mu_0} \otimes \omega_U = \mathcal{E}^{\mathrm{sub}}_{\mu_0+w}.$$

We can canonically lift a power of Hasse from $\mathbb{F}_p$ to $\mathbb{Z}_{(p)}$, and this allows us to define a map for every positive integer K

$$H^0(\mathcal{X}^{\mathrm{min}}, \mathcal{E}^{\mathrm{sub}}_{\mu_0+p^{K-1}(p-1)\cdot w}) \to H^0(\mathcal{X}^{\mathrm{ord,min}}, \mathcal{E}^{\mathrm{sub}}_{\mu_0} \otimes \mathbb{Z}/p^K\mathbb{Z})$$

$$f \mapsto f|_{\mathrm{ord}} / \tilde{\mathrm{Hasse}}.$$

The p-adic interpolation result then tells us that if we sum over all multiples of $p^{K-1}(p-1)$, then every p-adic automorphic form mod p^K has a preimage in the space of classical automorphic forms, i.e.

Proposition 6 (HLTT)**.** *For all r and for all K positive integers*

$$\bigoplus_{j=r}^{\infty} H^0(\mathcal{X}^{\mathrm{min}}, \mathcal{E}^{\mathrm{sub}}_{\mu_0+jp^{K-1}(p-1)\cdot w}) \twoheadrightarrow H^0(\mathcal{X}^{\mathrm{min,ord}}, \mathcal{E}^{\mathrm{sub}}_{\mu_0} \otimes \mathbb{Z}/p^K\mathbb{Z})$$

In GL_2, when one has q-expansions, the analogue is: If f is a weight k overconvergent eigenform with $k < 0$, then there exist classical eigenforms g_i of weight $k_i > 2$ such that

$$f \equiv g_i \bmod p^i \qquad \text{where } k_i = k + j(p-1)$$

Now we will use the theory of pseudocharacters and carefully chosen Hecke operators to get a representation associated to $\Pi(M)$ as a p-adic limit of representations $r_p(\Pi')$ for various classical $GU(n,n)$-automorphic representations Π'.

Unramified case

Here, there exist integral Hecke operators T_v (analogue of T_p in GL_2) inside $\mathbb{Z}[G(\mathbb{Z}_\ell)\backslash G(\mathbb{Q}_\ell)/G(\mathbb{Z}_\ell)]$ (here, $v \mid \ell$) which by Satake have eigenvalue on classical Π'

$$\mathrm{tr}\,\mathrm{rec}(\mathrm{BC}(\Pi'_v) \otimes |\det|^{(1-2n)/2})(\mathrm{Frob}_v) = \mathrm{tr}\, R_p(\Pi')^{ss}(\mathrm{Frob}_v),$$

because Π' is conjugate self-dual, and we already have LGC for such $R_p(\Pi')$.

The ramified case

The Bernstein center for GL_{2n} is

$$\mathfrak{Z}_{v,\mathbb{Q}_p} := \lim_{\leftarrow K} \mathcal{Z}(\mathbb{Q}_p[K \backslash \mathrm{GL}_{2n}(F_v)/K]),$$

It is commutative, and Bernstein showed the existence of various idempotents $e_{\mathfrak{B}}$ such that the image $\mathfrak{z}_v = e_{\mathfrak{B}} \mathfrak{Z}_{v,\mathbb{Q}_p}$ maps into the endomorphism algebras of the spaces of p-adic and classical automorphic forms we've dealt with thus far. In $\mathfrak{z}_v$, Chenevier provides us with

Hecke operators $T_{\sigma,v}$ inside the Bernstein center where $\sigma \in W_{F_v}$, whose eigenvalue on classical Π'

$$\operatorname{tr}\operatorname{rec}(\mathrm{BC}(\Pi'_v) \otimes |\det|^{(1-2n)/2})(\sigma) = \operatorname{tr}\mathrm{WD}(R_p(\Pi'))^{ss}(\sigma_v).$$

These Hecke operators live in a Hecke algebra defined over some finite extension of $\mathbb{Q}_p$, and we take scalar multiples if necessary to see these Hecke operators as $\mathbb{Z}_p$-endomorphisms of whichever H^0 space.

Pseudocharacters

Let

$$\mathcal{H}^p = \bigotimes_{\text{unr. } \ell} \mathbb{Z}[G(\mathbb{Z}_\ell)\backslash G(\mathbb{Q}_\ell)/G(\mathbb{Z}_\ell)] \otimes \bigotimes_{\text{ram. } \ell} \mathfrak{z}_{v,\mathbb{Z}_p},$$

and let $\mathbb{T}^p_\mu$ denote the image in $\operatorname{End}_{\mathbb{Z}_p}(H^0(\mathcal{X}^{\min}, \mathcal{E}^{\mathrm{sub}}_\mu))$. If μ satisfies the classicality condition, then by the existence of Galois representations, we see that there is a continuous pseudocharacter

$$T : G_F \to \mathbb{T}^p_\mu \quad \mathrm{Frob}_v \mapsto T_v \quad \sigma \mapsto T_{\sigma,v}.$$

If $\mathbb{T}^{\mathrm{ord},p}_{\mu_0}$ denotes the image of $\mathcal{H}^p$ in $\operatorname{End}_{\mathbb{Z}_p}(H^0(\mathcal{X}^{\min,\mathrm{ord}}, \mathcal{E}^{\mathrm{sub}}_{\mu_0} \otimes \mathbb{Z}/p^K\mathbb{Z}))$, then by the p-adic interpolation result, we can patch together a pseudocharacter

$$T^{\mathrm{ord}} : G_F \to \mathbb{T}^{\mathrm{ord},p}_{\mu_0} \quad \mathrm{Frob}_v \mapsto T_v \quad \sigma \mapsto T_{\sigma,v}.$$

For $\Pi(M)$, there is a map $\mathbb{T}^{\mathrm{ord},p}_{\mu_0} \to \overline{\mathbb{Q}}_p$ sending Hecke operators to their eigenvalues on

$\Pi(M)^U$, thus we then get

$$T^{\text{ord}}_{\pi(M)} : G_F \to \overline{\mathbb{Q}}_p \qquad \text{Frob}_v \mapsto \operatorname{tr}\operatorname{rec}(\text{BC}(\pi(M)_v \otimes |\det|^{(1-2n)/2}))(\text{Frob}_v)$$
$$\sigma \mapsto \operatorname{tr}\operatorname{rec}(\text{BC}(\pi(M)_v \otimes |\det|^{(1-2n)/2}))(\sigma)$$

Thus by the theory of pseudocharacters, we get a Galois representation

$$r_p(\pi(M)) : G_F \to \text{GL}_{2n}(\overline{\mathbb{Q}}_p)$$

satisfying for all unramified primes

$$\begin{aligned} \text{WD}(r_p(\pi(M))|_{G_{F_v}})^{ss} &\cong \operatorname{rec}(\text{BC}(\pi(M)_v) \otimes |\det|^{(1-2n)/2}) \\ &\cong \imath^{-1}\operatorname{rec}(\pi_v \otimes |\det|_v^{(1-n+M)/2}) \oplus (\imath^{-1}\operatorname{rec}(\pi_{^c v}|\det|_{^c v}^{(1-n+M)/2}))^{\vee,c}\epsilon_p^{1-2n}. \end{aligned}$$

since $BC(\pi(M)) = (\pi \otimes ||\det||^M) \oplus (\pi \otimes ||\det||^M)^{\vee,c}$.

Because we have constructed the above representations for all sufficiently large M, it is now just group theory to isolate the n-dimensional guys.