MSRI Hot Topics Workshop: Perfectoid Spaces and their Applications

Lubin-Tate Spaces II - Jared Weinstein

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Summary: This lecture continues from where the speaker left off in his previous lecture, where we constructed the Lubin-Tate space at infinite level as the base change of a mysterious "determinant" map. The main goal of this lecture is to show how we can give a concrete description of what this determinant map is. This is done by choosing coordinates and introducing quasi-logarithms, which lets us relate the mysterious map to an actual determinant and thus give an explicit formula in the coordinates. Finally, we talk about period maps and the geometry of Lubin-Tate space in light of these calculations.

We start by tying up a few loose ends from the previous talk. Let H_0/k be a p-divisible group; it has a Dieudonné module $M(H_0)$ that's free over $W(k)$ with rank equal to the height of H_0 and comes with endomorphisms F, V with $FV = p$ which satisfy $\dim_k M/FM = \dim H_0$.

Last time, tried to define $\bigwedge^n M(H_0)$. Can define $\bigwedge^n F$, but also need a V, so need that $\bigwedge^n F$ divides p, and that requires $\dim H_0 \leq 1$. If $\dim H_0 = 1$ and H_0 has height n, then we can actually form $\bigwedge^r H_0$ with height $\binom{n}{r}$ and dimension $\binom{n-1}{r-1}$.

From now on, H_0 is the thing we were deforming from last time, so it has height n, dimension 1, and be formal (i.e. connected). In this case F will be the matrix

$$F = \begin{bmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ p & & & \end{bmatrix}.$$

Then $\det F = p$, and $\bigwedge^n H_0 \cong \mu_{p^\infty}$.

The main result from last time was that we had a Cartesian diagram (where

$\eta = \mathrm{Spa}\ K$ for $K \supseteq W(k)$ a perfectoid field):

$$\begin{array}{ccc} \mathcal{M}_{H_0,\infty} & \longrightarrow & (\widetilde{H}^{\mathrm{ad}}_\eta)^n \\ \big\downarrow & & \big\downarrow{\scriptstyle \det} \\ \mathcal{M}_{\bigwedge^n H_0,\infty} & \longrightarrow & \widetilde{\Lambda^n H}^{\mathrm{ad}}_\eta . \end{array}$$

All of these are perfectoid spaces; recall that $\mathcal{M}_{\bigwedge^n H_0,\infty}$ is $V\mu_{p^\infty} \setminus \{0\}$, which is a non-locally-geometrically-connected space space. (Remark: If we used $\eta = \mathrm{Spa}(K_0, \mathcal{O}_{K_0})$ then $\mathcal{M}_{\bigwedge^n H_0,\infty}$ is a pre-perfectoid space that's a disjoint union $\coprod_{\mathbb{Z}} \mathrm{Spa}(K_\infty, \mathcal{O}_{K_\infty})$ for $K_\infty = K_0(\zeta_{p^\infty})^\wedge$. But if we base-change to something big, the space splits apart).

The only thing really mysterious in that diagram is the determinant map. Let's make that more explicit. We have

$$(\widetilde{H}^{\mathrm{ad}}_\eta)^n \cong \mathrm{Spf}\ \mathcal{O}[[X_1^{1/p^\infty}, \cdots, X_n^{1/p^\infty}]]^{\mathrm{ad}}_\eta,$$

$$\widetilde{\Lambda^n H}^{\mathrm{ad}}_\eta \cong \mathrm{Spf}\ \mathcal{O}[[T^{1/p^\infty}]]^{\mathrm{ad}}_\eta.$$

Then det corresponds to a function $\delta(X_1, \ldots, X_n)$ between these rings. In this explicit setup, the bottom map corresponds to a map $\mathcal{O}_{K_0}[[T^{1/p^\infty}]] \to \mathcal{O}_{K_\infty}$ given by $T \mapsto \lim_{m\to\infty}(1 - \xi_{p^m})^{p^m}$ on the 0-th copy of the thing in the disjoint union in the domain.

Some explicit formulas: Let R be a f-semiperfect k-algebra. Then we said last time that we had

$$\widetilde{H}_0(R) \cong \mathrm{Hom}_{F,\varphi}(M(H_0), B^+_{\mathrm{crys}}(R)).$$

Consider the case $n = 1$; the isomorphism above is one $\widetilde{\mu}_{p^\infty}(R) \cong B^+_{\mathrm{crys}}(R)^{\varphi=p}$. Moreover, explicitly we have

$$\widetilde{\mu}_{p^\infty}(R) = \varprojlim_{x \mapsto x^p} (1 + \mathrm{Nil}(R)) \subseteq R^\flat,$$

and we map an element x to $\log[x]$.

Now consider a general H_0, a 1-dimensional formal group of height n. Let $H/\mathcal{O}_{K_0}$ be a lift (where $\mathcal{O}_{K_0} = W(k)$). As a formal group, this has a logarithm $\log_H : H \otimes K_0 \cong \widehat{\mathbb{G}}_a$, and in fact this logarithm determines H fully. Now want to describe the Dieudonné module, via quasi-logarithms. Given $g(T) \in K_0[[T]]$, let

$$\delta g(x, y) = g(X +_H Y) - g(X) - g(Y),$$

so δg measures the failure of g to be an additive homomorphism. Now set

$$\mathrm{QLog}(H) = \frac{\{g \in TK_0[[T]] : dg, \delta g \text{ Integral}\}}{T\mathcal{O}_{K_0}[[T]]}.$$

Fact: $M(H_0) \cong \mathrm{QLog}(H)$, and is spanned (after inverting $1/p$) by $v_m = \log_H(T^{p^m})$ for $0 \le m \le n-1$. Then F acts by $Fv_i = v_{i+1}$ (at least for $i < n-1$) and this determines what V does. Remark: $\mathrm{QLog}(H)$ really only depends on H_0, but the choice of lift H gives a canonical element $\log_H$.

Given $g \in \mathrm{QLog}(H)$, can evaluate on $\widetilde{H}$. Let (R, R^+) be an affinoid K_0-algebra. Then have map $\widetilde{H}(R^+) \to R$, which we also denote g; an element of $\widetilde{H}(R^+)$ is a compatible sequence (x_i), and we map it to $\lim_{m\to\infty} p^m g(x_m)$. We claim that this g is actually a homomorphism $\widetilde{H}_\eta^{\mathrm{ad}} \to \mathbb{G}_a$. (Well-definedness follows from compatibility of the sequence, and that it's a homomorphism follows from δg being integral so the failure of the original g being a homomorphism gets pushed off to 0).

Now let (R, R^+) be a perfectoid affinoid. Recall that we had an isomorphism

$$\widetilde{H}_0(R^+/p) = \widetilde{H}(R^+) \cong \mathrm{Hom}_{F,\varphi}(M(H_0), B^+_{\mathrm{crys}}(R))$$

$$\cong \left(M(H_0)^* \otimes B^+_{\mathrm{crys}}(R^+)\right)^{F\otimes\varphi}.$$

Now, we have a morphism $\theta : B^+_{\mathrm{crys}}(R^+) \to R$, so composing the above isomorphism with $1 \otimes \theta$ gives a morphism

$$\widetilde{H}_0(R^+/p) \to M(H_0)^* \otimes R = \mathrm{Hom}(M(H_0), R).$$

We call this morphism qlog_{H_0}; it's given by taking x to the homomorphism $M(H_0) \to R$ given by $g \mapsto g(x)$. So we get an adic-space morphism $\mathrm{qlog}_{H_0} : \widetilde{H}_\eta^{\mathrm{ad}} \to M(H_0)^* \otimes \mathbb{G}_a \cong \mathbb{G}_a^n$. Taking a sum of n copies of these gives a map $(\widetilde{H}_\eta^{\mathrm{ad}})^n \to \mathbb{G}_a^{n^2}$. Claim that our mysterious determinant map fits into a commutative diagram

$$\begin{array}{ccc} (\widetilde{H}_\eta^{\mathrm{ad}})^n & \xrightarrow{\mathrm{qlog}_{H_0}} & \mathbb{G}_a^{n^2} \\ \downarrow{\scriptstyle \det} & & \downarrow{\scriptstyle \det} \\ \widetilde{\textstyle\bigwedge^n H}_\eta^{\mathrm{ad}} & \xrightarrow[\log_{\bigwedge^n H_0}]{} & \mathbb{G}_a. \end{array}$$

So if we can determine the quasi-logarithms explicitly we have an explicit formula for the determinant map we want! To do this, choose coordinates on H and $\bigwedge^n H$ so that

$$\log_H(T) = T + \frac{T^{p^n}}{p} + \frac{T^{p^{2n}}}{p^2} + \cdots$$

$$\log_{\bigwedge^n H}(T) = T + (-1)^{n-1}\frac{T^p}{p} + \frac{T^{p^2}}{p^2} + \cdots$$

Proposition 1. *Let $\log_{\widetilde{H}}$ denote the composite of the natural map $\widetilde{H}_\eta^{\mathrm{ad}} \to H_\eta^{\mathrm{ad}}$ with the map $\log_H : H_\eta^{\mathrm{ad}} \to \mathbb{G}_a$. Moreover, recall that $\widetilde{H}_\eta^{\mathrm{ad}}$ is the perfectoid unit* *disc* Spf $\mathcal{O}_{K_0}[[T^{1/p^\infty}]]^{\mathrm{ad}}_\eta$*; viewing it as that, the map* $\log_{\widetilde{H}}$ *is given by*

$$x \mapsto \sum_{n\in\mathbb{Z}} \frac{x^{p^i}}{p^i}.$$

Proof of this proposition is some yoga on what the map $\widetilde{H}^{\mathrm{ad}}_\eta \to H^{\mathrm{ad}}_\eta$ is in terms of limits.

Since we know what the logarithm is and a basis for the space of quasi-logarithms, can finally find an explicit expression for the determinant map det, written as $\delta(X_1, \ldots, X_n)$ on the level of coordinates.

Proposition 2. *We have*

$$\delta(X_1, \ldots, X_n) = \sum_{\substack{(a_1,\ldots,a_n)\in\mathbb{Z} \\ \sum a_i = n(n-1)/2 \\ \{a_i \bmod n\} = \mathbb{Z}/n\mathbb{Z}}} \varepsilon(a) X_1^{p^{a_1}} \cdots X_n^{p^{a_n}}$$

where the sum is in $\bigwedge^n H$ *and* $\varepsilon(a)$ *is the sign of* $i \mapsto a_{i+1} \bmod n$*.*

We can prove this by plugging the formula into the diagram and seeing it commutes, plus a bit of extra checking.

Period maps: Recall that we have $\mathrm{qlog}_{H_0} : \widetilde{H}^{\mathrm{ad}}_\eta \to M(H_0)^* \otimes \mathbb{G}_a$. Lubin-Tate space at infinite level can be described as a functor given by

$$\mathcal{M}_{H_0,\infty}(R, R^+) = \Big\{(X_1, \ldots, X_n) \in \widetilde{H}(R^+)^n : \delta(X_1, \ldots, X_n) \in V\mu_{p^\infty} \setminus \{0\}\Big\}.$$

(a slight lie, since it's really the sheafification of this). However, saying that the element $\delta(X_1, \ldots, X_n)$ lives in the Tate module (i.e. that it's torsion) is the same as saying that its logarithm is zero. So we can further describe this set as

$$\Big\{(X_1, \ldots, X_n) \in \widetilde{H}(R^+)^n : X_i \text{ lin. indep. } /\mathbb{Q}_p, \log\delta(X_1, \ldots, X_n) = 0\Big\}.$$

But by our commutative diagram, we have

$$\log\delta(X_1, \ldots, X_n) = \det(\mathrm{qlog}_{H_0}(X_1, \ldots, X_n)).$$

So for such elements, $\mathrm{qlog}(X_1), \ldots, \mathrm{qlog}(X_n)$ are linearly dependent in $M(H_0)^* \otimes \mathbb{G}_a$. The span of these elements is a plane in $\mathbb{P}^{n-1}$ (and the map $\mathcal{M}_{H_0,\infty} \to \mathbb{P}^{n-1}$ is the Gross-Hopkins period map π_{GH}). The vector of linear relations lies in $\mathbb{P}^{n-1}$ minus all $\mathbb{Q}_p$-rational hyperplanes (since we said we were linearly independent over $\mathbb{Q}_p$), which is the Drinfeld space Ω. The map to Ω is called the Hodge-Tate period map π_{HT}.

So we have $\mathcal{M}_{H_0,\infty}$, with an action of $\mathrm{GL}_n(\mathbb{Q}_p) \times D^\times$. Then π_{GH} is a "quotient by $\mathrm{GL}_n(\mathbb{Q}_p)$" which goes to $\mathbb{P}^{n-1} = \mathbb{P}(M(H_0))$ (which has a $D^\times$-action). Similarly π_{HT} is a "quotient by $D^\times$" and goes to Ω (which has a $\mathrm{GL}_2(\mathbb{Q}_p)$-action).

Conjectures about the geometry of $\mathcal{M}_{H_0,\infty}$. Let $\eta = \mathrm{Spa}(C, \mathcal{O}_C)$, and consider the map $\mathcal{M}_{H_0,\infty} \to V\mu_{p^\infty} \setminus \{0\}$. Let $\mathcal{M}^{\cdot}_{H_0,\infty}$ be one fiber of this. The idea is that $H^*(\mathcal{M}_{H_0,\infty}, \overline{\mathbb{Q}}_\ell)$ realizes the local Langlands correspondence. If $\pi \subseteq H^*(\mathcal{M}_{H_0,\infty}, \overline{\mathbb{Q}}_\ell)$ is a supercuspidal, then $\pi = \mathrm{Ind}_{\mathscr{K}}^{\mathrm{GL}_n(\mathbb{Q}_p)}(\tau)$ where $\mathscr{K}$ is open and compact modulo the center, and τ is a finite-dimensional representation.

Let $(\mathbb{P}^{n-1})^{\text{special}}$ be the locus in $\mathbb{P}^{n-1}$ where the stabilizer in $\mathrm{PGL}_n(\mathbb{Q}_p)$ is nontrivial. (In terms of p-divisible groups, the special ones are the ones with extra automorphisms). The non-special locus $(\mathbb{P}^{n-1})^{\text{nonspecial}}$ in fact lives inside of Ω. Let

$$\mathcal{M}_{H_0,\infty}^{\text{nonspecial}} = \pi_{HT}^{-1}[(\mathbb{P}^{n-1})^{\text{nonspecial}}].$$

Conjecture 3. *A connected component of* $\mathcal{M}_{H_0,\infty}^{\text{nonspecial}}$ *can be covered by affinoids* U *with* $\dim H^i(U, \mathbb{Q}_\ell) < \infty$. *(True for* $n = 2$ *by checking).*

Fantasy: One could even hope that the tilt of one of these connected components is locally p-finite in the perfectoid sense.