Geometric Nonlinear Dispersive PDE's 3

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Recall the equations:

Wave Maps:

$$u : \mathbb{R}^{n+1} \to (M, g) \subset \mathbb{R}^m$$

$$\Box = -S(u) \cdot \partial^\alpha u \partial_\alpha u$$

With critical exponents given by $s_c = \frac{n}{2}$ for $n \geq 2$.

Max-Klein-Gordon:

$$\left\{ \begin{array}{l} \Box_A \phi = 0 \\ \partial^\beta F_{\alpha\beta} = 0 \end{array} \right\} \Box A = A\nabla A + A^3$$

Critical exponents given by $s_c = \frac{n}{2} - 1$ for $n \geq 4$.

Yang-Mills:

$$D^\beta F_{\alpha\beta} = 0, \quad F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta]$$

Small data missing items:

1. Function spaces S, N. Want to capture the dispersive properties of the wave equation. To do this the following are utilized:
   (a) energy estimates
   (b) Strichartz estimates
   (c) $X^{s,b}$ spaces
   (d) null frame spaces (related to the Lorenz invarance from first lecture)
   (e) U^p, V^p spaces
2. In these spaces, prove bilinear, multilinear structures. In particular, this requires an understanding of the resonance structure of the wave equation. i.e. how do waves interact? Small vs large angle interactions as well as null structure should be considered.
3. Parametrics construction.

Large data problem

Question: Given large data (u_0, u_1) in the critical Sobolev norm, is there a global soln in S? i.e. no blow-up and modified scattering exists.

Usually the answer to these questions is "no" (two out of three equations result in "no"). There are a few obstructions:

1. Self-similar solutions (have the form $u(t, x) = t^\gamma v\left(\frac{x}{t}\right)$). You have these solutions for all of the equations, but not in the energy critical case for (WM), (YM), (MKG).
2. You can get solutions that don't decay, e.g. solitons. In the energy critical case you get solitons for (WM) and (YM) but not (MKG).

Threshold Conjecture: In the energy critical case, we have global well-posedness if $E < E_0$ (ground state energy, i.e. the smallest energy of a soliton).

Remark 0.1 *Once you have one soliton, you have at least a one-parameter family of solitons due to rescaling, translating, or Lorenz transformation. One can often construct blowup solutions where the blowup occurs along these solitons. Since there are no solitons for (MKG) we expect global solutions from this conjecture.*

Theorem 0.1 (Sterbenz, Tataru) *For the wave map equation*

Remark 0.2 *A less general theorem was proven independently by Tao for H^n and Krieger-Schlag for H^2. (MKG) was studied by S. J. Oh and Tataru as well as Krieger-Luhmann. (YM) is still open.*

Question: For all of these problems, what is the effect of a small data result on this Threshold Conjecture.

Suppose we have a point of blowup. For the wave equation, this point would be influenced by the data within a backwards light cone (since finite speed of propagation). On some slice of the light cone, if we have small energy, then we can extend the slice out a little bit which would allow us to extend the data past the point of blowup. Thus if we have a small data result but no large data result, then we must have energy concentration in a light cone. If t_0 is the time of blowup, then

$$\lim_{t \to t_0} E(u)(t) > E_{\text{small data}}$$

Methods in the large data problem

Induction on energy (Bourgain):
For each energy E there exists some constant $F(E)$ such that if $E(u) \leq E$ then u is global and

$$\|u\|_S \leq F(E). \tag{1}$$

Alternatively, we can consider a continuity argument. Let $\varepsilon = \{E \text{ for which (1) holds}\}$. To show ε is open we use perturbation theory. To show ε is closed is where Bourgain's induction of energy comes into play.

To see how we use Bourgain's argument, imagine the statement holds for E and we try to have this holds for $E+c$. The solutions will have some size $F(E)$ at E. When we do perturbation theory, the constants will depends on this size. Thus c is dependent on $F(E)$ which is problematic as $c(F(E))$ might be bounded.

Now if this analysis fails, we can find an $E_{\min}$ where the method fails. We can find solutions u_n such that $E(u_n) \to E_{\min}$ and $\|u_n\|_{S^1} \to \infty$. Bourgain argues that eventually in the sequence (u_n) we must some very specific types of concentration. Then show that this type of concentration cannot occur very many places which is done via Moravets estimates.

Remark 0.3 *The Kenig-Merle method is an improvement on the induction of energy method.*

Energy Dispersive Solutions: this method has a different way of splitting the analysis from the Kenig-Merle method. This method seems to work quite well for wave equations. We will now outline this method. Let's start with the (WM) equation. Recall $H^1(\mathbb{R}^2) \subset L^\infty$ dyadically. Take a solution u and write it's L-P decomposition $u = \sum u_k$. Next we look at $\sup_k \|u_k\|_{L^\infty} = \|E\|_{ED}$ where $\|\cdot\|_{ED}$ is the energy dispersed norm.

Theorem 0.2 *For each E, there $\exists \varepsilon(E), F(E)$ such that if $E(u) \leq E, \|u\|_{ED} \leq \varepsilon(D)$, then $\|u\|_s \leq F(E)$.*

We will now point out some of the difficulties that arise when proving this theorem.

We don't get rid of induction on energy because proof of this theorem is induction on energy. We assume the statement holding at E and want to show that it holds at $E + c$. We also need that c depends on E not $F(E)$. A small sketch:

Suppose $E(u) = E+c$ and we chop off the top frequences of u so that u reduces to size E: $E(\tilde{u}) = E$ where $\tilde{u} = P_{<k}(u)$.

So what happens is u lives in the space where energy is $E + c$ and $\tilde{u}$ lives in space where energy is E. The low frequencies for u and $\tilde{u}$ are the same, but not the high frequencies.

$$\begin{aligned} &\|u\|_{ED} \leq \varepsilon \quad \text{ want } \Rightarrow \|u\|_s \leq F(E+c) \\ &\|\tilde{u}\|_{ED} \leq \varepsilon(E) \Rightarrow \|\tilde{u}\|_s \leq F(E) \end{aligned}$$

We want show that if $\|u\|_{ED}$ then $\|\tilde{u}\|_{ED}$. To do this we need notice that if ε is sufficiently small, we can use $\|u\|_S \leq F(E+c)$ to show $\|\tilde{u}\|_{ED}$. Then we show that $\|\tilde{u}\|_S \leq F(E) \Rightarrow \|u\|_s \leq F(E)$. Lastly, we want c to depends on only E. To do this we use the divisibility of the S-norm. Chop the time interval into pieces where $\|\tilde{u}\|_{S(E)} \leq E$ on each interval. For the energy critical problem, the energy is conserved. On each interval we get a nice bound and at the end of the argument we add up the pieces to get a bound dependent on E.