Dynamical and Spectral Properties of Mathematical Billiards

Alfonsio Sorrentino

Intertwine between dynamics & geometry

Mathematical Billard (Serge Tabachnikov textbook)

domain in $\mathbb{R}^2$
point mass travels in straight lines
angle of incidence = angle of reflection when it hits boundary

Can be extended: to Riemannian manifold $(M, \partial M, g)$
to higher dimension – reverse normal velocity, keep tangential velocity

very different dynamical systems can result from different geometries:
- Polygonal
- Dispersive (Sinai)

Convex Billiards (Birkhoff)

convex condition plays the role of the twist condition

$\Omega \subset \mathbb{R}^2$ bounded strictly convex domain in $\mathbb{R}^2$
$\partial\Omega$ boundary C^r └ curvature at every point is strictly positive
$\ell = |\partial\Omega|$, $\gamma : \mathbb{R}/\ell\mathbb{Z} \to \mathbb{R}^2$ $\quad \kappa(\gamma(s)) > 0$

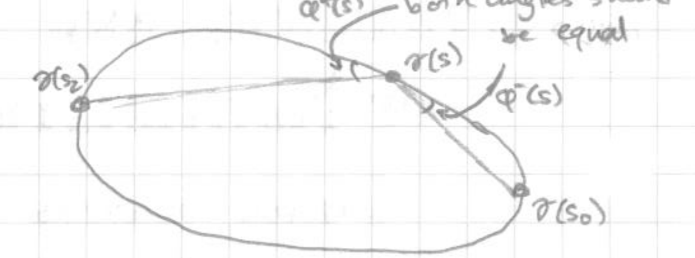
in between bounces, nothing interesting happens → just take a map on the boundary

$$B = B_\Omega : \underbrace{\mathbb{R}/\ell\mathbb{Z}}_{\text{location along boundary}} \times \underbrace{(0,\pi)}_{\text{angle}} \to \mathbb{R}/\ell\mathbb{Z} \times (0,\pi)$$

strange phenomenon happens when $r = 2$
in finite time, it can start going along the boundary

Properties
- $B \in C^{r-1}(\mathbb{R}/\ell\mathbb{Z} \times (0,\pi))$
  continuous extension to $\{\varphi = 0\}$ & $\{\varphi = \pi\}$
- B is a symplectic map $\quad \omega = \sin\phi \, d\phi \wedge ds$
  $B_* \omega = \omega$
  B is an explicit symplectic (twist) map → it has a generating function

$$h(s_0, s_1) = -\|\gamma(s_0) - \gamma(s_1)\| \quad \begin{cases} \partial_1 h(s_0, s_1) = \cos\varphi_0 \\ \partial_2 h(s_0, s_1) = -\cos\varphi_1 \end{cases}$$



How do you shoot a billiard from one side to the one, bouncing once?

Pick some $s \in (s_0, s_1)$

Consider the length of this polygonal

$L(s) = h(s_0, s) + h(s, s_1)$ this should be minimizer

$L'(s) = -\cos\varphi^-(s) + \cos\varphi^+(s)$

orbit $\longleftrightarrow$ critical configuration

$\{(s_m, \varphi_m)\}_{m\in\mathbb{Z}}$ $\qquad$ $\{s_m\}_{m\in\mathbb{Z}}$

$$\mathcal{A}(\{s_m\}) = \underbrace{\sum_{m\in\mathbb{Z}} h(s_m, s_{m+1})}_{\text{to make this finite, fix endpoints, \# bounces}}$$

Do global minima exist? Aubry-Mather Thm

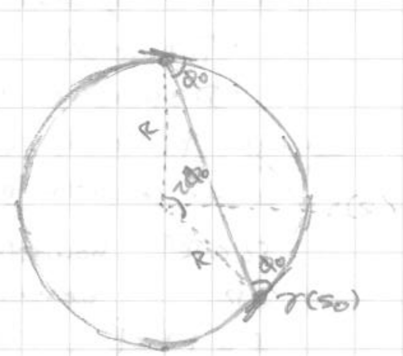
Can we find?

Periodic Orbits, Invariant Curves, Integrability

Ex <u>Circular Billiard</u>

angle stays the same.

$B(s_0, \varphi_0) = (s_0 + 2R\varphi_0, \varphi_0)$



Phase space is a cylinder

- $\varphi_0 = \frac{p}{q}\pi$ periodic orbits
- $\varphi_0 \notin \mathbb{Q}$ quasiperiodic orbits

The orbits remain tangent to a smaller circle

take an orbit (s_m, φ_m). lift to universal cover $(\tilde{s}_m, \varphi_m)$

define <u>rotation number</u> $\dfrac{\tilde{s}_m - \tilde{s}_0}{m\ell} \to \rho \pmod{\mathbb{Z}}$

$\rho = 1/5$

Rotation number for periodic orbits

we only need to look at rotation numbers between $(0, \frac{1}{2}]$ since the motion can be reversed

ρ (Periodic Orbit) = Winding Number/Period

$\rho = 2/5$

<u>Thm</u> (Birkhoff)

$\forall\ \frac{p}{q} \in (0, \frac{1}{2}] \cap \mathbb{Q}$

there exist at least two periodic orbits

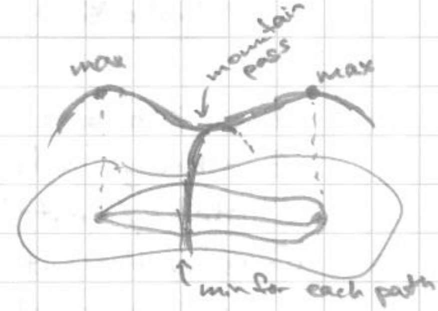
<u>Sketch of Proof</u> $\Sigma = \{ \{x_m\} \in \mathbb{R} : x_{m+p} = x_m + p,\ x_i \neq x_{i+1} \forall i \}$ ← polygon of these points

$L : \overline{\Sigma} \to [0, \infty)$ length of polygon

close the set

$$L(\{x_m\}) = -\sum_{m=0}^{p-1} h(x_m, x_{m+1})$$

look at the argmax - not unique - can rotate
will be in the set
argmin is in the boundary
to find second, use a mtn pass
take 2 max's - look at all paths between
- find min of each path - max over paths



What is the measure of the set of initial data that correspond to periodic orbits? Ivrii (Conjecture) = 0.

Length spectrum of Ω

$\mathcal{L}(\Omega) = \mathbb{N}^+ \{\text{length of all p.o.}\} \cup \mathbb{N}^+ \ell$

connected to spectrum of laplacian

$$\text{Spec}(\Omega) = \begin{cases} \Delta u = \lambda u & \Omega \\ u = 0 & \partial\Omega \end{cases}$$

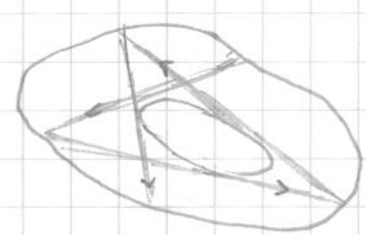
Andersson - Melrose $\omega(t) = \sum_{\lambda_i \in \text{spec}(\Omega)} \cos(t\sqrt{-\lambda_i})$ wave trace

Can you hear the shape of a drum?

sing sup $\omega \subseteq \pm \mathcal{L}(\Omega) \cup \{0\}$
= generically

Results
De Simoi - Kaloshin, Wei, Deform of $\mathbb{Z}^2$ symmetry
" " , Fleser? nearly circular Ω

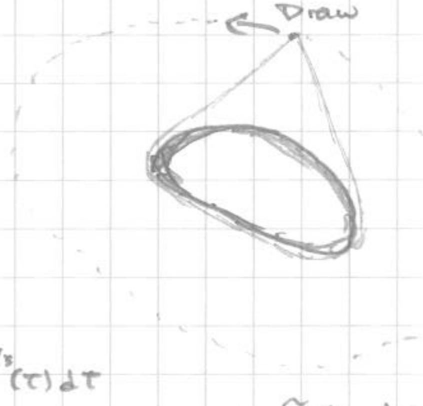
Def Caustic
Γ caustic for Ω if an orbit is tangent to it for each bounce



Def Convex Caustic
Γ is a convex caustic if Γ is C^1, simple, closed
it bounds a convex region

Existance of Caustics
① to make a billiard with one predetermined caustic

take a string of fixed length, wrap it around the caustic & draw the shape

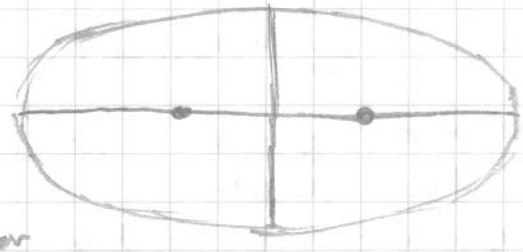


② infinitely many caustics
Lazutkin: change variables

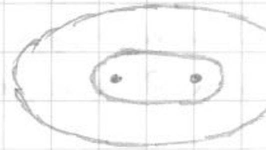
$$\begin{cases} x = C\int_0^s \chi^{2/3}(\tau)\,d\tau \\ y = 4C \sin\frac{\varphi}{2}\, \chi^{-1/3} \end{cases} \rightsquigarrow \tilde{B}(x,y) = (x+y, y) + O(y^3)$$

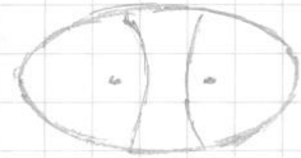
so as long as you have enough deriv for KAM,
we get invariant tori for Diophantine $\to$ ∞ly many

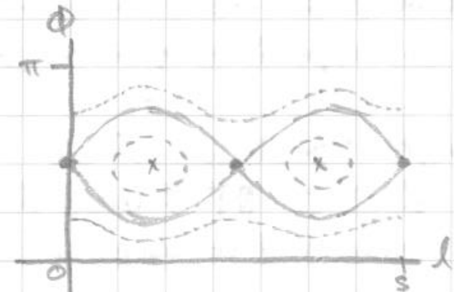
③ which billiards are integrable?
Foliated by caustics
<u>Elliptic Billiards</u>



Period 2 orbits - semimajor, semiminor
Go through a focus - approach semimajor
stable/unstable manifolds
Cross center outside foci - never cross
semimajor between foci → caustic
Cross center inside foci - stay within
hyperboloid → caustic







<u>Conjecture</u> (Birkhoff - Porski?)
the only integrable billiards are ellipse.

<u>Thm</u> (Bialy & Wojtkowski)
If the phase space is completely foliated by homotopic nontrivial invariant curve ⇒ Ω is a disk.

Perturbative version of the conjecture
DeSimoi - Kaloshin 2016 Ω close to a disk
Kaloshin - Sorrentino 2018 Ω close to an ellipse
if the billiard have caustics with rotation number $\frac{1}{q}$, $q \geq 3$ ⇒ Ω ellipse
periodic orbits are the first to break if it's not integrable