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NORMAL FORMS AND KAM THEORY IN CELESTIAL MECHANICS

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2 issues for problems in celestial mechanics
- how to find a good model
- how to implement the theory in this model

Prologue:

1846 – LeVerrier – discrepency between theory & observation of Uranus's orbit suggests existence of Neptune – used perturbation theory

Galle – discovered Neptune

Delaunay – spent 20 yrs computing precise motion of the moon

$H = h + \varepsilon f$ – perturbation for moon's motion

expand in Fourier series in angle, Taylor series in action – expansion is 22 papers long

this theory is still used by NASA, ESA.

Perturbation theory vs. KAM theory

different requirements: non-resonance vs. Diophantine

$|\omega \cdot k| \neq 0$ $\qquad |\omega \cdot k| > |k|^{-\tau}$

both theories lead to efficient computation algorithm.

Aim of classical perturbation theory:

construct a canonical transformation to push perturbation to higher orders of ε

theory is constructive – proof shows you how to construct new normal form.

$$H(J,\varphi) = Z(J) + \varepsilon R(J,\varphi) \rightsquigarrow H'(J',\varphi') = \underbrace{Z(J') + \varepsilon \overline{R}(J')}_{=:Z'(J')} + \varepsilon^2 R'(J',\varphi')$$

look for a generating function χ for this transformation $\qquad \chi \sim O(\varepsilon)$

Split $\varepsilon R \to \varepsilon \overline{R} + \varepsilon \widetilde{R}$, $\overline{R}$ = average, $\widetilde{R}$ = remainder

To find χ, solve $\varepsilon \widetilde{R} + \{Z, \chi\} = 0$. Normal form.

Expand both χ, R in a series, set coefficients equal.

$$\hat{\chi}_k = \varepsilon \frac{\hat{R}_k(J')}{i\,\omega(J')\cdot k}$$ for coeff of the series

where we need $\omega(J')\cdot k \neq 0$.

A resonance occurs when $\omega(J)\cdot k = 0$

eg. same side of moon always faces earth.

Can we do perturbation theory here?

Separate out the resonant part of R.

$R = \overline{R} + R_{res} + \widetilde{R}(I,\varphi)$

Example Space Debris

distinguish regions above earth into:

low earth orbit – feel atmospheric drag – dissipative

medium earth orbit – no atmospheric drag – effects of sun, moon, non-spherical earth — GPS – 2:1 resonance

geostationary orbit – 1:1 resonance with rotation of earth

Goals: gravitational resonance with Earth's rotation

secular resonance with sun/moon – longer

mechanism for onset of chaos & disposal orbits?

Models: Cartesian – most commonly used – easy to include all terms

Hamiltonian – gives you more structure

Graveyard orbits – ones that are safe for other satellites

Delaunay Variables:

actions – $L = \sqrt{\mu a}$, $G = L\sqrt{1-e^2}$, $H = G\cos i$

↑ major axis ↑ eccentricity ↑ angle relative to equatorial plane

Hamiltonian integrators can be much faster than Cartesian

Hamiltonian approach automatically tells you the key features of the problem

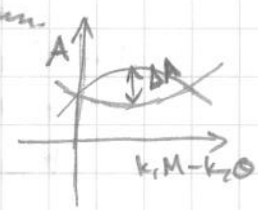
naturally untangles multiple time scales for the resonances.

First-Order Normal Form

resonance only depends on one angle → integrable – like pendulum.

easy to compute amplitude of pendulum's libration.

⇒ very different stability for GPS vs. GEO



Long-term dynamics of GEO:

forced equilibrium – equilibrium associated with normal form

→ analytic form for eccentricity, angle i

in the original coords, this trajectory is a 5-dim torus.

KAM Theory

extend KAM to conformally symplectic (dissipative) system

this is a very singular perturbation → only a few attractors

in celestial problems, we often have to include dissipation

dumbell asteroid – not entirely rigid → internal dissipation

Hénon: in order for KAM to apply, ε must be really small $\frac{m}{M_\odot} < 10^{-48}$

Conservative & Dissipative Standard Maps

useful exercise for students: numerically look at behavior as ε changes

dissipative – one invariant → breaks to periodic orbit

Most of our theory here has been conservative.
But - dissipative effects often relevant for celestial mechanics.
For KAM, we also need a family of conformally symplectic maps.
invariance condition involves a drift
Also need a good approx. initial guess of solution.

Conclusion → ∃ attractive quasi-periodic solution with this frequency

From the proof, you can write down an algorithm to numerically construct invariant tori.
Only needs $O(N)$ storage, $O(N \log N)$ operations - cheap

implements Newton's method from KAM machinery
-starting from initial guess.

Start with simple models - toy models
standard map
write proofs, build numerical techniques
can get numerical results extremely close to when the torus eventually breaks

luckily, most real celestial problems are far from largest ε before torus breaks

Spin - Orbit for a non-spherical shape
Dissipation from tidal torque
Conservative case - use KAM to show stability of orbit of moon
for true parameters for moon, you get bounding invariant tori
moon's orbit isn't a KAM torus, but it's trapped between them
Dissipative case - existance theorem, but no concrete estimates yet

Conservative 3-Body Problem - Sun, Jupiter, asteroid 12 Victoria
can give KAM estimates that are consistent with astronomical observations
degenerate Hamiltonian, but satisfies alternative criterion by Arnold
computer-assisted proof shows KAM tori holds → complete stability

Still many open questions
use normal forms to compute families of space debries
designing desposal orbits for space debries
compute stable/unstable manifolds for the invariant tori that have been found.