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Name: Tony Feng **Email/Phone:** tonyfeng@stanford.edu

Speaker's Name: Claire Voisin

Talk Title: Stable birational invariants

Date: 2 / 6 / 19 **Time:** 10 : 30 am / pm (circle one)

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STABLE BIRATIONAL INVARIANTS

CLAIRE VOISIN

1. Unramified cohomology

Let $X/\mathbf{C}$ be smooth. We have $f\colon X_{an} \to X_{Zar}$ and we defined $\mathcal{H}^i(A) = R^i f_* A$,

$$H^i_{nr}(X;A) := H^0(X_{Zar}, \mathcal{H}^i(A)).$$

We would like to explain why this is a birational invariant.

We recall the *Gersten-Quillen resolution* of $\mathcal{H}^i(A)$. For all W irreducible and reduced, define

$$H^\ell(\mathbf{C}(W);A) := \varinjlim_{U\subset W} H^\ell_B(U;A).$$

This is a constant sheaf on W. If W contains a divisor D, we have a residue map

$$H^i(\mathbf{C}(W);A) \to H^{i-1}(\mathbf{C}(D);A)$$

(The definition is slightly tricky.)

The Gersten-Quillen resolution is

$$0 \to \mathcal{H}^i(A) \to H^i(\mathbf{C}(X);A) \to \bigoplus_{\mathrm{codim}(D)=1} H^{i-1}(\mathbf{C}(D);A) \to \ldots \to \bigoplus_{\mathrm{codim}(Z)=i} H^0(\mathbf{C}(Z);A) \to 0. \tag{1.1}$$

Why is it a complex? It basically amounts to saying taking residues in one order is negative of the residues in the other order.

Theorem 1.1 (Bloch-Ogus)**.** *This is an acyclic resolution of $\mathcal{H}^i(A)$.*

This is a deep and difficult fact.

We have a spectral sequence $E_2^{p,q} = H^p(X_{Zar}; \mathcal{H}^q(A))$ converging to the analytic cohomology. The Bloch-Ogus theorem shows that

$$E_2^{0,i} = H^i_{nr}(X;A) = \ker\left(H^i(\mathbf{C}(X);A) \xrightarrow{\mathrm{res}} \bigoplus_D H^{i-1}(D;A)\right). \tag{1.2}$$

Corollary 1.2. *For $U \subset X$, $H^i_{nr}(X;A) \to H^i_{nr}(U;A)$ is injective and an isomorphism if the codimension of $X-U$ has codimension at least 2.*

Proof. Neither term in (1.2) changes. □

This implies the birational invariance of $H^i_{nr}(X;A)$, for smooth projective X.

Other corollaries:

(1) $E_2^{p,q} = 0$ for $p > q$.

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(2) The Bloch-Ogus formula.

Theorem 1.3 (Bloch-Ogus)**.** $CH^k(X)/alg.eq \cong H^k(X_{Zar}; \mathcal{H}^k(\mathbf{Z}))$.

Proof. Use the resolution (1.1). We're looking at the last term, which is a direct sum over cycles of codimension k of $H^0(\mathbf{C}(Z); \mathbf{Z})$, modulo the sum over cycles W of codimension $k-1$ of $H^1(\mathbf{C}(W); \mathbf{Z})$. This is the same as the relation of algebraic equivalence. □

We have a filtration $E_\infty^{p,q}$ on $H_B^*(X; \mathbf{Z})$ with $p+q = 2k, p \leq q$. In particular $E_\infty^{k,k}$ is the sub. We just found $E_2^{k,k}$ in terms of the Chow group. Since the differentials leaving $E_2^{k,k}$ are all 0, we have $E_2^{k,k} \twoheadrightarrow E_\infty^{k,k} \subset H_B^{2k}(X; \mathbf{Z})$; this is the cycle class map.

2. Chow decomposition of the diagonal

Let X be a smooth variety of dimension n over an algebraically closed k. Choose $x \in X$ a point of degree 1.

Definition 2.1. We say that X has a *Chow decomposition of the diagonal* if

$$\Delta_X = X \times x + Z \in CH^n(X \times X) \tag{2.1}$$

where $Z = \sum n_i Z_i$ is such that $pr_1 : Z_i \to X$ does not dominate.

Equivalently, there exist a proper closed $D \subset X$ such that Z_i is supported $D \times X$.

Consider the action of correspondences $P \in CH^n(X \times X)$, which induces $P_* \colon CH^k(X) \to CH^k(X)$ by

$$P_*(z) = pr_{2*}(pr_1^* P \cap z)$$

with adjoint

$$P^*(z) = pr_{1*}(pr_2^* P \cap z).$$

Lemma 2.2. *If X has a decomposition of the diagonal, then $CH_*(X) = \mathbf{Z}x$.*

Proof. Consider the action of (2.1): we get

$$z = (\deg z)x + 0.$$

□

Over $\mathbf{C}$ there is a sort of converse, due to Bloch-Srinivas.

Theorem 2.3 (Bloch-Srinivas)**.** $CH_0(X) = \mathbf{Z}$ *implies* (2.1) *after tensoring with* $\mathbf{Q}$.

Lemma 2.4. *If X has a Chow decomposition of the diagonal, then for all $L \supset k$ then $CH_0(X_L) = \mathbf{Z}x_L$.*

Proof. Just take a base change of the decomposition and apply the same argument. □

Definition 2.5. X has *universally trivial* CH_0 if it has the property that for all $L \supset k$ then $CH_0(X_L) = \mathbf{Z}x_L$.

Proposition 2.6 (Auel-Colliot-Thélène-Parimala)**.** *X has a Chow decomposition of the diagonal if and only if X has universally trivial CH_0-group.*

Proof. Assume $CH_0(X_L) = \mathbf{Z}x_L$ for all $L \supset K$. Consider $L = K(X)$. By hypothesis, the generic point $\eta_L \in X_L(L)$ is identified with x_L in $CH_0(X_L)$. You then spread this out, viewing L as the colimit of functions over Zariski open subsets of X. From this you deduce that there exists $U \subset X$ such that $\Delta(X)|_{U\times X} = U \times x \in CH^n(U \times X)$. Using the localization exact sequence, this gives a decomposition of the diagonal. □

Proposition 2.7 (V.)**.** *If X has a decomposition of the diagonal modulo algebraic equivalence, then it has a Chow decomposition of the diagonal.*

3. Cohomological decomposition of the diagonal

Definition 3.1. We say that X has a *cohomological decomposition of the diagonal* if

$$[\Delta_X] = [X \times x] + [Z] \in H^{2n}_B(X \times X) \tag{3.1}$$

where $[Z] = \sum n_i[Z_i]$ with the property that $pr_1 : Z_i \to X$ does not dominate.

Equivalently, there exist a proper closed $D \subset X$ such that Z_i is supported $D \times X$.

If you have a Chow decomposition, you get a cohomological one by taking the cycle class.

Remark 3.2. Having a Chow decomposition of the diagonal is a stably birationally invariant property for X smooth projective.

Proof. Use that $\mathbf{P}^r$ has a Chow decomposition of the diagonal, because

$$\Delta_{\mathbf{P}^r} = \sum pr_1^* h^i \cdot pr_2^* h^{r-i}$$

for h the hyperplane class.

Take $0 \in \mathbf{P}^r$. First restrict $\Delta_{X\times\mathbf{P}^r}$ to $X \times 0 \times X \times \mathbf{P}^r$, and then project to $\Delta_X \subset X \times X$. Applying this to a decomposition of $\Delta_{X\times\mathbf{P}^r}$ gives a decomposition of $\Delta_{X\times\mathbf{P}^r}$.

Then we do the birational invariance. Suppose X is birational to Y. Resolve by X' mapping to both. We have $(\varphi,\varphi)_*\Delta_{X'} = \Delta_X$ and $(\varphi,\varphi)^*\Delta_X$ agrees with $\Delta_{X'}$ on the open subset where φ is injective, hence agrees up to something on the locus where φ is not an isomorphism. □

Theorem 3.3. *(1) If X has a cohomological decomposition of the diagonal, then $H^3_B(X;\mathbf{Z}) = 0$ and $Z^4(X) = 0$.*

(2) If X has a Chow decomposition of the diagonal, then $H^i_{nr}(X;A) = 0$ for all $i > 0$.

(3) If X in positive characteristic has a Chow decomposition of the diagonal, Totaro has shown that $H^0(\Omega^k_{X/K}) = 0$.

Proof. (1) $[\Delta_X] = [X \times x] + [Z]$ where $[Z]$ is supported on $D \times X$. We let $\widetilde{D} \to D$ be the desingularization in such a way that Z lifts to $\widetilde{Z} \in CH^{n-1}(\widetilde{D} \times X)$. Let $j : \widetilde{D} \to X$ be the composition.

We can thus write

$$[\Delta_X] = [X \times x] + (j, \mathrm{Id})_*[\widetilde{Z}].$$

Hence for all $\alpha \in H_B^\ell(X; \mathbf{Z})$ we will get

$$\alpha = \underbrace{[X \times x]^*\alpha}_{=0 \text{ if } \ell>0} + j_*([\widetilde{Z}]^*\alpha)$$

and $[\widetilde{Z}]^*\alpha$ factors through $H_B^{\ell-2}(\widetilde{D}; \mathbf{Z})$.

If $\alpha \in H_B^3(X; \mathbf{Z})_{\mathrm{tors}}$ then $\alpha = j_*([\widetilde{Z}]^*\alpha) = 0$.

If $\alpha \in \mathrm{Hdg}^4(X; \mathbf{Z})$ then $\alpha = j_*([\widetilde{Z}]^*\alpha)$ which factors through $\mathrm{Hdg}^2(\widetilde{D}; \mathbf{Z})$, and is algebraic.

For (2) and (3), the argument is similar but using a different cycle class – the Bloch-Ogus cycle class for (2), and for case (3) the de Rham cycle class of a correspondence. □