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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Michael McNulty Email/Phone: mmcnulty@math.ucr.edu / 6099821994

Speaker's Name: Rafe Mazzeo

Talk Title: Geom microlocal analysis lecture 1

Date: 9 / 3 / 19 Time: 2 : 30 am / pm (circle one)

Please summarize the lecture in 5 or fewer sentences: This lecture introduced the construction of parametrices on certain manifolds with fame geometry at infinity.

CHECK LIST

(This is **NOT** optional, we will **not** pay for **incomplete** forms)

☑ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.

☑ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.

- Computer Presentations: Obtain a copy of their presentation
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☑ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)

☑ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

MSRI LECTURES ON GEOMETRIC MICROLOCAL ANALYSIS LECTURE 1

LECTURER: RAFE MAZZEO

Abstract. Rough notes for lectures on geometric microlocal analysis at the MSRI introductory workshop in Fall 2019.

- <u>Plan for this Lecture Series</u>:
  1. What is geometric microlocal analysis? Generalities plus a case study.
  2. Further details about the case study
  3. Further examples.
- <u>Aphorisms</u>:
  1. When in doubt, compactify.
  2. If you are still in doubt, blow something up.
  3. Smoothness is not what it seems.
- We want to study PDEs on space which are:
  1. Noncompact, complete manifolds with tame geometry at infinity like $\mathbb{R}^n$ and $\mathbb{H}^n$.
  2. Singular spaces like cones, edges, stratified spaces, etc.
  3. Spaces or operators which are degenerating; adiabatic limits, neck stretching, geometric gluing, etc.
- Examples of singular spaces: level sets of Morse functions; algebraic varieties; compactification of moduli spaces; compactification of locally symmetric spaces.
- <u>Focus on elliptic operators</u>: $L = \sum_{|\alpha| \leq m} a_\alpha(z) D_z^\alpha$. The principal symbol equals $\sigma_m(L)(z, \xi) = \sum_{|\alpha|=m} a_\alpha(z)\xi^\alpha$, and ellipticity means that this is invertible when $\xi \neq 0$.
- <u>A priori estimates vs. parametrices</u>:
  - The classical Sobolev estimates for an elliptic operator $\|u\|_{H^{s+m}} \leq (\|Lu\|_{H^s} + \|u\|_0)$ (strictly speaking, the norm on the left should be over a domain which is smaller than the one used on the right;
  - A parametrix for L is an approximate inverse, namely a pseudodifferential operator $G \in \Psi^{-m}$ such that $LG = I - R_1$, $GL = I - R_2$ where the two operators $R_1, R_2 \in \Psi^{-\infty}$ are 'residual'.

    Existence of a pseudodifferential parametrix and knowledge of its mapping properties imply the Sobolev estimates:

  * $Lu = f$ implies $u - R_1 u = Gf$, which gives $\|u\|_{H^{s+m}} \leq C(\|f\|_{H^s} + \|u\|_0)$ since $G : H^s \to H^{s+m}$ is bounded and $R_1 : H^t \to H^{t'}$ is bounded for all t, t'
- One can use the a priori estimates plus a bit of functional analysis to deduce the existence of a (generalized) inverse for L, i.e., an inverse up to finite rank errors (the projects onto the kernel and cokernel). However, this does not tell you the structure of this generalized inverse.
- Our goal: a global theory of parametrices:
  - L a differential operator on manifold M, G a parametrix for L with $G(z, z') \in \mathcal{D}'(M \times M)$ its Schwartz kernel.
    * We care about the geometric structure of G
  - After compactifying, things become interesting at the boundary
- Singular integral operators $\frac{F(\frac{z}{|z|})}{|z|^n}$, $\int_{S^{n-1}} F = 0$.
  - Oscillatory integral representation of ΨDOs
  - Melrose and collaborators led to Schwartz kernels
- Hadamard Parametrix Construction: (due to Friedlander)
  - Given L, find the parametrix G, $G \sim \sum_{j=0}^{\infty} G_j$ with $LG_0 = I - R_0$
  - If $L = \sum_{|\alpha|=m} a_\alpha(z_0) D_z^\alpha$; $G_0(z, \tilde{z}) = \mathcal{F}^{-1}\left(\frac{1}{\sum_{|\alpha|=m} a_\alpha(z_0)\xi^\alpha}\right)$
  - For χ a suitable cutoff function, this is

$$\int e^{i(z-\tilde{z})\cdot\xi} \frac{\chi(\xi)}{\sum_{|\alpha|=m} a_\alpha(z_0)\xi^\alpha} d\xi$$

  - For instance, if $m = 2$ and $L = \Delta$, then $\mathcal{F}^{-1}\left(\frac{1}{\xi^2}\right) = \frac{1}{|z - \tilde{z}|^{n-2}}$
  - $LG_0 = I +$ error and $G_0(z, \tilde{z}) \sim a_0(z) d(z, \tilde{z})^{m-n}$
  - $LG_0 = I - R_0(z, \tilde{z})$, $R_0(z, \tilde{z}) \sim d(z, \tilde{z})^{+n}$
  - $L(G_0 + ... + G_N) = I - R_N$, $R_N \sim d(z, \tilde{z})^{-n+N+1} + C^\infty(M \times M)$.
  - Want an asymptotic sum: $\tilde{G} \sim \sum G_j$, $L\tilde{G} = I - \tilde{R}$; $R \in C^\infty(M \times M)$.
  - In this construction, any M works but if M is open or singular, then $\tilde{R} \in C^\infty(M \times M)$ is not necessarily a compact operator.
  - $\tilde{G}^t L = I - \tilde{R}^t$, $\tilde{R}^t : \mathcal{E}'(M) \to C^\infty(M)$ does not improve growth.
- Turn our attention to $\mathbb{H}^n$:
  - Metric in half-space model:

$$\frac{dx^2 + dy^2}{x^2} \tag{1}$$

  - Metric in Poincaré disk:

$$\frac{4|dz|^2}{(1 - |z|^2)^2} \tag{2}$$

- Metric in Klein model:
$$\frac{4(\sum z_j dz_j^2)}{(1-|z|^2)^2} + \frac{4|dz|^2}{1-|z|^2} \tag{3}$$
- Laplacian in half-space:
$$\Delta_g = x^2\partial_x^2 + (2-n)x\partial_x + x^2\Delta_y \tag{4}$$
which degenerates at $x = 0$.
- Laplacian in Poincaré:
$$\Delta_g = (1-|z|^2)^2\Delta_z - 2(2-n)\sum z_i\partial_{z_i} \tag{5}$$
- (M, g) conformally compact with $g = \rho^{-2}\bar{g}$, $\rho = 0$ on ∂M, $d\rho \neq 0$.
- (r, θ) on $\mathbb{H}^n$ with r distance from origin and $\theta \in S^{n-1}$,
$$g = dr^2 + \sinh^2 r d\theta^2 \tag{6}$$
and
$$\Delta = \sinh^{1-n} r\partial_r(\sinh^{n-1} r\partial_r) + \frac{1}{\sinh^2 r}\Delta_\theta \tag{7}$$
- Want to solve
$$G'' + (n-1)\frac{\cosh r}{\sinh r}G' = 0,\ r > 0 \tag{8}$$
- Thus, either $G \sim r^0, r^{2-n}$
- With $\rho = e^{-r}$, we have $G \sim \rho^{n-1}$ as $\rho \to 0$, i.e., $r \to \infty$.
- At the boundary of $M \times M$ compactified we have nice expansions with $G \sim r^{2-n}$ toward the diagonal.
- What happens at the corners of the diagonal?
- Blow up the corner along the diagonal
- Dilation invariant: $G(x, y, \tilde{x}, \tilde{y}) = G(\lambda x, \lambda y, \lambda\tilde{x}, \lambda\tilde{y})$
- How to localize?
- $\mathbb{H}^n \setminus \Omega$, $\Delta + V$, $V \in C_0^\infty$, define $\tilde{G} = \tilde{\chi}_1 G_{in}\chi_1 + \tilde{\chi}_2 G_{out}\chi_2$ where $\chi_1, \tilde{\chi}_1$ compactly supported in U_1 and $\chi_2, \tilde{\chi}_2$ compactly supported in U_2 where $\Omega \subset U_1 \subset U_2$.
- $G_{out} = G_{\mathbb{H}^n}$, G_{in} =standard local ΨDO.
- Want to obtain local parametrices and glue them together.