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Note: These solutions are a work in progress; comments, references, etc. are appreciated. Most references and figures can be found with the problem statements.

Problem 1. (a) A unit cube is cut in half by a saw cut that is perpendicular to its long diagonal. What is the shape of the cross section of the cut?

(b) Generalize to n dimensions.

Discussion: Holding the cube with its long diagonal vertical, we see that the bisecting cut intersects 6 edges (that zig-zag up and down when the cube is held in this position), and that the cross section is therefore a regular polygon with 6 vertices, i.e., a hexagon.

In n dimensions, we can take the intersection to be $C \cap H$ where $C := \{x \in \mathbf{R}^m : |x_i| \leq 1, \text{ for all } i\}$ is the unit cube, centered at the origin, in $\mathbf{R}^n$, and $H := \{x \in \mathbf{R}n : \sum x_i = 0\}$ is the sum-zero hyperplane. For even n, this is the convex hull of the n-tuples that have $n/2$ 1's and the same number of -1's. For odd n, the vertices are all n-tuples that have a single 0, and $(n-1)/2$ 1's and the same number of -1's.

Problem 2. You owe your friend 62 cents, but have only a (fair) dollar coin. Devise a sequence of fair coin flips which has the property that he wins the dollar coin exactly 62% of the time.

Discussion: Expand 62/100 in binary

$$62/100 = .1\overline{00111101011100001010}$$

Use the sequence of coin flips to determine the binary expansion of a number between 0 and 1. As soon as this sequence differs from the expansion of 62/100, the coin-flipping can cease. Depending on whether the random sequence is larger or smaller than 62/100, the loser pays the winner \$1. This idea can be generalized in the obvious way to any desired probability between 0 and 1.

John H. Conway has suggested another view of this same sequence of coin flips, which evades calculating the binary expansion of 62/100 in advance. Instead, he keeps track of the residual value of the bet as long as it remains unresolved. Since .62 exceeds 1/2, if the first coin flip is zero, the event ends. The 62 cent value is one half of \$1.24, which can be partitioned into two parts: an immediate one dollar payoff if the coin flip is heads, plus 24 cents, which is the residual value of the bet if the first coin flip is tails. This line of reasoning then continues as follows:

$$
\begin{aligned}
62 &= (100+24)/2\\
24 &= (0+48)/2\\
48 &= (0+96)/2\\
96 &= (100+92)/2\\
92 &= (100+84)/2\\
84 &= (100+68)/2\\
68 &= (100+36)/2\\
36 &= (0+72)/2\\
72 &= (100+44)/2\\
44 &= (0+88)/2\\
88 &= (100+76)/2\\
76 &= (100+52)/2\\
52 &= (100+4)/2\\
4 &= (0+8)/2\\
8 &= (0+16)/2\\
16 &= (0+32)/2\\
32 &= (0+64)/2\\
64 &= (100+28)/2\\
28 &= (0+56)/2\\
56 &= (100+12)/2\\
12 &= (0+24)/2
\end{aligned}
$$

24 returns to the second line of this sequence.

The possible immediate payoffs, of course, are the bits of the binary expansion of 64/100.

Problem 3. Suppose that n players shoot at bins labeled 1, 2, 3, ... and hit bin k with probability 2^{-k}. Let p_n be the probability that the highest bin hit is only hit once. Prove that p_n is not a monotone function of n and that in fact the limit

$$\lim_{n\to\infty} p_n$$

does not exist.

Discussion: Although even fairly serious numerical experiments will suggest that the limit exists, in fact it doesn't. The probability p_n turns out to be asymptotic to a periodic function of $\log_2(n)$; the constant term is $1/2\log(2) \simeq .71$, and the magnitude of all other fourier coefficients is less than 10^{-4}. This curious result (by now) occurs in many places in the literature, including the references added to the Athreya et al article in the problem statement (to several articles by Eisenberg et al), as well as articles by Prodinger and others (e.g., "How to select a loser," , *Discrete Mathematics*, **120** (1993) 149-159).

Problem 4. Let $a(n)$ be the number of digits in 2^n, in its usual decimal expansion, that are bigger than or equal to 5. Evaluate

$$\sum_{n=1}^{\infty} \frac{a(n)}{2^n}.$$

Discussion: Write the k-th digit (coefficient of 10^k) of the decimal expansion of 2^n in the "base 5" form

$$e_{k,n} + 5f_{k,n}, \qquad e_{k,n} \in \{0, 1, 2, 3, 4\},\ f_{k,n} \in \{0, 1\}.$$

We are being asked to evaluate

$$S := \sum_{n\geq 1, k\geq 0} \frac{f_{k,n}}{2^n}.$$

From the definitions we have

$$2^n = \sum_{k\geq 0} (e_{k,n} + 5\, f_{k,n}) 10^k.$$

Multiplying by 2 gives

$$2^{n+1} = \sum_{k\geq 0} (2e_{k,n} + 10\, f_{k,n}) 10^k = \sum_{k\geq 0} 2e_{k,n} 10^k + \sum_{k\geq 0} f_{k,n} 10^{k+1}$$

from which we deduce that for $n \geq 1$ and $k \geq 1$ we have

$$e_{k,n+1} + 5\, f_{k,n+1} = 2e_{k,n} + f_{k-1,n}.$$

Dividing by 2^{n+1} and summing over all positive k and n (and noting that $e_{k,1} = 0$ for positive k) gives

$$\sum_{k\geq 1, n\geq 1} \frac{e_{k,n}}{2^n} + 5(S - T) = \sum_{k\geq 1, n\geq 1} \frac{e_{k,n}}{2^n} + (1/2)S$$

where $T := \sum_{n\geq 2} f_{0,n}/2^n$. The first two sums cancel and we are left with

$$(9/2)S = 5T.$$

The 1's digits of 2^n, $n > 0$, form a periodic sequence $2, 4, 8, 6, 2, 4, 8, 6, \cdots$, i.e., $f_{0,n}$ is 0 if $n \equiv 1, 2 \bmod 4$ and is 1 if $n \equiv 3, 4 \bmod 4$. An exercise in geometric series shows that $T = 1/5$ and we find that $S = 2/9$.

Problem 5. A mathemagician has an audience select 5 numbers from the set $\{1, 2, \cdots, n\}$. (The case of $n = 52$ is of course of special interest.)

The mathemagician thinks carefully about the 5 numbers and then names 4 numbers, and asks an audience member to write those 4 numbers, in the order given, on a blackboard. (The audience contains no shills, i.e., people who collude with the mathemagician.)

The mathemagician then leaves the room and a partner, who has been oblivious to the proceedings so far, is summoned to enter the room through a different door. After announcing that "numbers naturally come in sets of five, so that the outlier can be determined from remaining four," she studies the four numbers on the board and correctly announces the fifth number in the original collection.

How can this be done when $n = 52$?

What is the maximum value of n for which this is possible, no matter which 5-element subset is chosen?

Devise practical algorithms for the two mathemagicians so that this trick could actually be performed.

Discussion: The assistant can choose any of 5 numbers to hide, and he can order the others in any of $4! = 24$ different ways, giving a total of 120 choices. The mathemagician sees 4 of the N possible numbers, leaving $N - 4$ possibilities. So it appears that no strategy can guarantee success unless $N \leq 124$. Here is a specific strategy which attains that goal:

For convenience, we assume the numbers range from 0 to 123. The assistant first computes the sum of the five selected numbers, modulo 5. Call this remainder s, where $0 \leq s < 5$. (We will use lower case letters for numbers with small upper bounds; upper case letters, for numbers with larger upper bounds.) The assistant then selects the s-th smallest of the five chosen numbers to become the hidden number, X. Next, the assistant divides $X - s$ by 5, obtaining a quotient Q and a remainder r, where $0|leQ < 24$ and $0 \leq r < 5$. The integer Q is then expanded into the factorial basis

$$Q = \sum_{i=1}^{3} a_i i!$$

This equation determines the coefficients a_1, a_2, and a_3, with $0 \leq a_i \leq i$. These coefficients are then used to select the appropriate ordering of the four numbers which will be written on the board. Specifically, let these visible numbers be ordered as V_3, V_2, V_1, V_0, where the ordering is selected in such a way that a_i is the number of subsequent V's (i.e., $V_{i-1}, \ldots, V_0$) which are less than V_i.

So for example, suppose the five numbers chosen by the audience are 12, 17, 49, 73, and 90. Modulo 5, their sum is 1. So the assistant chooses 17 as the number to be hidden. He next divides 17-1 by 5, to obtain a quotient $Q = 3$ and a remainder $r = 1$. Next, the assistant writes the quotient $Q = 3$ in the factorial basis as $3 = 0 \cdot 3! + 1 \cdot 2! + 1 \cdot 1!$. So he selects the least of the four candidates (12) as the first number to be written on the board, then the next-to-least of the remaining 3 (73) as the second number to be written on the board, and the next-to-least of the remaining 2 (90) as the third number to be written on the board. So the numbers on the board are 12, 73, 90, 49, in that order.

When the mathemagician sees the board, he reverses this computation to determine that $Q = 3$. In general, the unknown number, X, must satisfy $X - s = 5Q + r$. It follows that $5Q \leq X \leq 5Q+8$. In this particular case, seeing 12, 73, 90, 49, and calculating that $Q = 0{\cdot}3!+1{\cdot}2!+1{\cdot}1! = 3$, he concludes that $15 \leq Q \leq 23$. Since only one of the four visible numbers is less than 23, in this case the mathemagician can conclude that $s = 1$. This enables him to sharpen his bounds to $16 \leq X \leq 20$. Finally, since $X + 12 + 73 + 90 + 49$ is congruent to s congruent to 1 mod 5, he is able to deduce that X is congruent to 2, and hence that $X = 17$.

In other cases, when one or more of the visible numbers lies strictly between $5Q$ and $5Q+8$, it may not be possible to determine the value of s quite so quickly. However, there is always a unique solution for s, r, and X. For suppose that the solution is not unique, and that there are two solutions, X, s, r and X', s', r'. Then if the four visible numbers are denoted by A, B, C and D, mod 5 we have $A + B + C + D$ congruent to $s - X$ congruent to $s' - X'$, so that $r' = r$, and $s - X = s' - X'$. Without loss of generality, we may assume $s' > s$. But then $X' = X + s' - s$. There must also be $s' - s$ visible numbers in the interval strictly between X' and X, but there are only $s' - s - 1$ numbers in this interval.

With a moderate amount of practice, a skilled mathemagician can usually rather quickly finish the determination of X by trial and error.