Introduction to resolutions
Irena Swanson, MSRI, Lecture 1

A **complex**: $\cdots \to M_{i+1} \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \to \cdots$, where the M_i are groups (or left modules), the d_i are group (or left module) homomorphisms, and for all i, $d_i \circ d_{i+1} = 0$. Abbreviate as $C_\bullet$, $M_\bullet$ or $(M_\bullet, d_\bullet)$, et cetera.

$(M_\bullet, d_\bullet)$ is **bounded below** if $M_i = 0$ for all sufficiently small (negative) i;
it is **bounded above** if $M_i = 0$ for all sufficiently large (positive) i;
it is **bounded** if $M_i = 0$ for all sufficiently large $|i|$.

$(M_\bullet, d_\bullet)$ is **exact at the ith place** if $\ker(d_i) = \operatorname{im}(d_{i+1})$.

A complex is **exact** if it is exact at all places. (Also called **exact sequence**.)

A complex is **free** (resp. **flat**, **projective**, **injective**) if all the M_i are free (resp. flat, projective, injective).

An exact complex $0 \to M' \xrightarrow{i} M \xrightarrow{p} M'' \to 0$ is called a **short exact sequence**.

`http://www.reed.edu/~iswanson/MSRI11SwansonIntroResolutions.pdf`

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A cocomplex is a complex that is numbered in the opposite order: $C^\bullet$: $\cdots \to M^{i-1} \xrightarrow{d^{i-1}} M^i \xrightarrow{d^i} M^{i+1} \to \cdots$.

What do we do with complexes?

(1) Given a module, build some corresponding complexes $\longrightarrow$ resolutions of modules (free, projective, flat, injective, minimal).
(2) Applying functors to (exact) complexes $\longrightarrow$ Hom, tensor products, tensor products of two complexes.
(3) Koszul complex: as tensor product of complexes (two ways), and there is a definition via exterior powers.
(4) Kernels and images in a complex $\longrightarrow$ homology.
(5) When are resolutions finite?
(6) When is a complex exact?
(7) What invariants (ranks?) appear in exact complexes?

Suppose that $\cdots \to F_{i+1} \xrightarrow{d_{i+1}} F_i \xrightarrow{d_i} F_{i-1} \to \cdots \longrightarrow F_1 \xrightarrow{d_1} F_0 \to M \to 0$ is exact and all the F_j are free (respectively projective, flat) modules over R, and M is an R-module. Then we call both the exact complex above, as well as the complex $\cdots \to F_{i+1} \xrightarrow{d_{i+1}} F_i \xrightarrow{d_i} F_{i-1} \to \cdots \to F_1 \xrightarrow{d_1} F_0 \to 0$, a **free (resp. projective, flat) resolution** of M.

Every R-module has a free resolution, and hence a projective resolution. If for $F_n = 0$ for some/all large n, we say that M has a **finite free/projective resolution**. The smallest n such that $F_{n+1} = F_{n+2} = \cdots$ is called **the length of this resolution**. The **free/projective dimension of** M is the smallest such n (it could be ∞). This is denoted as $\mathrm{pd}(M)$.

Minimal resolutions

Suppose that $0 \to M \to I^0 \to I^1 \to I^2 \to I^3 \to \cdots$ is exact, where each I_j is an injective R-module. This cocomplex, as well as $0 \to I^0 \to I^1 \to I^2 \to I^3 \to \cdots$ are called an **injective resolution** of M.

Ways to make complexes from a given one

$$C_\bullet = \qquad \cdots \to C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} C_{n-2} \to \cdots$$

(1) **Hom from** M: $\mathrm{Hom}_R(M, C_\bullet)$, with $\mathrm{Hom}(M, d_n) = d_n \circ \_$:

$$\cdots \to \mathrm{Hom}(M, C_n) \xrightarrow{\mathrm{Hom}(M,d_n)} \mathrm{Hom}_R(M, C_{n-1}) \xrightarrow{\mathrm{Hom}(M,d_{n-1})} \mathrm{Hom}_R(M, C_{n-2})$$

(2) **Hom to** M: $\mathrm{Hom}_R(C_\bullet, M)$ is a cocomplex, with $\mathrm{Hom}(d_n, M) = \_ \circ d_n$:

$$\cdots \to \mathrm{Hom}(C_{n-1}, M) \xrightarrow{\mathrm{Hom}(d_n,M)} \mathrm{Hom}_R(C_n, M) \xrightarrow{\mathrm{Hom}(d_{n+1},M)} \mathrm{Hom}_R(C_{n+1}, M$$

(3) **Tensor product $C_\bullet \otimes_R M$:**

$$C_\bullet \otimes_R M : \qquad \cdots \to C_n \otimes_R M \xrightarrow{d_n \otimes \mathrm{id}} C_{n-1} \otimes_R M \xrightarrow{d_{n-1} \otimes \mathrm{id}} C_{n-2} \otimes_R M \to \cdots.$$

(We say that M is **flat** if $C_\bullet \otimes M$ is exact for every exact complex $C_\bullet$.)

(4) **Tensor product of complexes:** Let $K_\bullet = \cdots \to K_n \xrightarrow{e_n} K_{n-1} \xrightarrow{e_{n-1}} K_{n-2} \to \cdots$ be another complex. The tensor product of complexes of $C_\bullet$ and $K_\bullet$ yields a kind of a bicomplex, as follows:

$$\begin{array}{ccccccc}
\downarrow & & \downarrow & & \downarrow & & \\
C_n \otimes K_{m+1} & \to & C_{n-1} \otimes K_{m+1} & \to & C_{n-2} \otimes K_{m+1} & \to \\
\downarrow & & \downarrow & & \downarrow & & \\
C_n \otimes K_m & \to & C_{n-1} \otimes K_m & \to & C_{n-2} \otimes K_m & \to \\
\downarrow & & \downarrow & & \downarrow & & \\
C_n \otimes K_{m-1} & \to & C_{n-1} \otimes K_{m-1} & \to & C_{n-2} \otimes K_{m-1} & \to \\
\downarrow & & \downarrow & & \downarrow & &
\end{array}$$

(Total complex along 45° angle) $G_n = \sum_i C_i \otimes K_{n-i}$, $g_n : G_n \to G_{n-1}$ defined on $C_i \otimes K_{n-i}$ as $d_i \otimes \mathrm{id}_{K_{n-i}} + (-1)^i \mathrm{id}_{C_i} \otimes e_{n-i}$, where the first summand is in $C_{i-1} \otimes K_{n-i}$ and the second in $C_i \otimes K_{n-i-1}$.

(Here we show that the total complex of the tensor product of complexes is a complex)
Recall $(C_\bullet, d_\bullet) \otimes (K_\bullet, e_\bullet)$ is:

$$\begin{array}{ccccccc}
\downarrow & & \downarrow & & \downarrow & & \\
C_n \otimes K_{m+1} & \longrightarrow & C_{n-1} \otimes K_{m+1} & \longrightarrow & C_{n-2} \otimes K_{m+1} & \longrightarrow \\
\downarrow & & \downarrow & & \downarrow & & \\
C_n \otimes K_m & \longrightarrow & C_{n-1} \otimes K_m & \longrightarrow & C_{n-2} \otimes K_m & \longrightarrow \\
\downarrow & & \downarrow & & \downarrow & & \\
C_n \otimes K_{m-1} & \longrightarrow & C_{n-1} \otimes K_{m-1} & \longrightarrow & C_{n-2} \otimes K_{m-1} & \longrightarrow \\
\downarrow & & \downarrow & & \downarrow & &
\end{array}$$

Total complex: $G_n = \sum_i C_i \otimes K_{n-i}$, $g_n : G_n \to G_{n-1}$ restricted to $C_i \otimes K_{n-i}$ is $d_i \otimes \mathrm{id}_{K_{n-i}} + (-1)^i \mathrm{id}_{C_i} \otimes e_{n-i}$.
This new construction is still a complex:

$$\begin{aligned}
g_{n-1} \circ g_n|_{C_i \otimes K_{n-i}} &= g_{n-1}(d_i \otimes \mathrm{id}_{K_{n-i}} + (-1)^i \mathrm{id}_{C_i} \otimes e_{n-i}) \\
&= d_{i-1} \circ d_i \otimes \mathrm{id}_{K_{n-i}} + (-1)^{i-1} d_i \otimes e_{n-i} \\
&\qquad + (-1)^i d_i \otimes e_{n-i} + (-1)^i (-1)^i \mathrm{id}_{C_i} \otimes e_{n-i-1} \circ e_{n-i} \\
&= 0.
\end{aligned}$$

Koszul complexes

Let R be a commutative ring, M a left R-module, and $x \in R$. The **Koszul complex** of x and M is

$$
\begin{array}{lccccccccc}
K_\bullet(x;M): & & 0 & \to & M & \xrightarrow{x} & M & \to & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 1 & & 0 & &
\end{array}
$$

If $x_1, \ldots, x_n \in R$, then the **Koszul complex** $K_\bullet(x_1, \ldots, x_n; M)$ of $x_1, \ldots, x_n$ and M is the total complex of $K_\bullet(x_1, \ldots, x_{n-1}; M) \otimes K_\bullet(x_n; R)$, defined inductively.

$K_\bullet(x_1, x_2; M)$ explicitly: From

$$\begin{pmatrix} 0 & \to & \underset{1}{M} & \xrightarrow{x_1} & \underset{0}{M} & \to & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & \to & \underset{1}{R} & \xrightarrow{x_2} & \underset{0}{R} & \to & 0 \end{pmatrix}$$

we get the total complex

$$0 \to \underset{1}{M} \otimes \underset{1}{R} \xrightarrow{\begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}} \underset{1}{M} \otimes \underset{0}{R} \oplus \underset{0}{M} \otimes \underset{1}{R} \xrightarrow{\begin{bmatrix} x_1 & x_2 \end{bmatrix}} \underset{0}{M} \otimes \underset{0}{R} \to 0$$

$K_\bullet(x_1, x_2; M)$ explicitly: From

$$\left(\begin{array}{ccccccc} 0 & \to & M & \xrightarrow{x_1} & M & \to & 0 \\ & & 1 & & 0 & & \end{array}\right) \otimes \left(\begin{array}{ccccccc} 0 & \to & R & \xrightarrow{x_2} & R & \to & 0 \\ & & 1 & & 0 & & \end{array}\right)$$

we get the total complex

$$\begin{array}{ccccccccc} 0 & \to & \underset{1\quad 1}{M \otimes R} & \xrightarrow{\begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}} & \underset{1\quad 0}{M \otimes R} \oplus \underset{0\quad 1}{M \otimes R} & \xrightarrow{\begin{bmatrix} x_1 & x_2 \end{bmatrix}} & \underset{0\quad 0}{M \otimes R} & \to 0 \end{array}$$

The following is $K_\bullet(x_1, x_2, x_3; R)$:

$$0 \to R \xrightarrow{\begin{bmatrix} x_3 \\ -x_2 \\ x_1 \end{bmatrix}} R^3 \xrightarrow{\begin{bmatrix} -x_2 & -x_3 & 0 \\ x_1 & 0 & -x_3 \\ 0 & x_1 & x_2 \end{bmatrix}} R^3 \xrightarrow{\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}} R \to 0.$$

Towards a more functorial construction of Koszul complexes

- The **n−fold tensor product**: $M^{\otimes 0} = R$; $M^{\otimes(n+1)} = M^{\otimes n} \otimes M$.
- The **nth exterior power of a module** M:

$$\wedge^n M = \frac{M^{\otimes n}}{\langle m_1 \otimes \cdots \otimes m_n : m_1, \dots, m_n \in M, m_i = m_j \text{ for some } i \neq j \rangle}.$$

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The image in $\wedge^n M$ of $m_1 \otimes \cdots \otimes m_n \in M^{\otimes n}$ is written as $m_1 \wedge \cdots \wedge m_n$. Since $0 = (m_1 + m_2) \wedge (m_1 + m_2) = m_1 \wedge m_1 + m_1 \wedge m_2 + m_2 \wedge m_1 + m_2 \wedge m_2 = m_1 \wedge m_2 + m_2 \wedge m_1$, we get that for all $m_1, m_2 \in M$, $m_1 \wedge m_2 = -m_2 \wedge m_1$.

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THUS: if $e_1, \ldots, e_m$ form a basis of R^m, then $\wedge^n R^m$ is **generated** by $B = \{e_{i_1} \wedge \cdots \wedge e_{i_n} : 1 \leq i_1 < i_2 < \cdots < e_{i_n} \leq m\}$. Actually, B is a basis for $\wedge^n R^m$.

Thus $\wedge^n R^m \cong R^{\binom{m}{n}}$.

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Thus $\wedge^n R^m \cong R^{\binom{m}{n}}$.
For any elements $x_1, \ldots, x_m \in R$ define a complex $G_\bullet(x_1, \ldots, x_n; R)$ as

$$0 \to \wedge^m R^m \to \wedge^{m-1} R^m \to \wedge^{m-2} R^m \to \cdots \to \wedge^1 R^m \to \wedge^0 R^m \to 0,$$

where $e_{i_1} \wedge \cdots \wedge e_{i_n} \mapsto \sum_{j=1}^n (-1)^{j+1} x_j \, e_{i_1} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_n}$.

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Facts:
- $G_\bullet(x; R) = K(x; R)$.
- $G_\bullet(x_1, \ldots, x_{n-1}; R) \otimes_R G_\bullet(x_n; R) \cong G_\bullet(x_1, \ldots, x_n; R)$.
- Thus $G_\bullet(x_1, \ldots, x_n; R)$ equals $K_\bullet(x_1, \ldots, x_m; R)$.

Homology, and special homologies

The **nth homology group (or module)** of $C_\bullet$ is $H_n(C_\bullet) = \frac{\ker d_n}{\operatorname{im} d_{n+1}}$.

Cohomology of a cocomplex $C^\bullet$ is $H^n(C^\bullet) = \frac{\ker d^n}{\operatorname{im} d^{n-1}}$.

- **Tor:** If M and N are R-modules, and if $F_\bullet$ is a projective resolution of M, then $\operatorname{Tor}_n^R(M,N) = H_n(F_\bullet \otimes N)$. You should know/prove that $\operatorname{Tor}_n^R(M,N) \cong \operatorname{Tor}_n^R(N,M)$, or in other words, that if $G_\bullet$ is a projective resolution of N, then $\operatorname{Tor}_n^R(M,N) \cong H_n(M \otimes G_\bullet)$.

- **Ext:** If M and N are R-modules, and if $F_\bullet$ is a projective resolution of M, then $\operatorname{Ext}_R^n(M,N) = H^n(\operatorname{Hom}_R(F_\bullet, N))$. You should know/prove that $\operatorname{Ext}_R^n(M,N) \cong H^n(\operatorname{Hom}_R(M, I_\bullet))$, where $I_\bullet$ is an injective resolution of N.

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Determine exactness/homology of a (part of a) complex ...

Definition 1 *A* **map of complexes** *is a function* $f_\bullet : (C_\bullet, d) \to ({C_\bullet}', d')$*, where* $f_\bullet$ *restricted to* C_n *is denoted* f_n*, where* f_n *maps to* C'_n*, and such that for all* n*,* $d'_n \circ f_n = f_{n-1} \circ d_n$*. We can draw this as a commutative diagram:*

$$\begin{array}{ccccccc}
\cdots \to & C_{n+1} & \overset{d_{n+1}}{\to} & C_n & \overset{d_n}{\to} & C_{n-1} & \to \cdots \\
 & \downarrow f_{n+1} & & \downarrow f_n & & \downarrow f_{n-1} & \\
\cdots \to & C'_{n+1} & \overset{d_{n+1}}{\to} & C'_n & \overset{d_n}{\to} & C'_{n-1} & \to \cdots
\end{array}$$

The kernel and the image of a map of complexes are naturally complexes. Thus we can talk about **exact complexes of complexes**.

Let $f_\bullet : C_\bullet \to {C_\bullet}'$ *be a map of complexes. Then we get the induced map* $f_* : H(C_\bullet) \to H({C_\bullet}')$ *of complexes.*

(Snake Lemma) Assume that the rows represent exact (parts of) complexes and the vertical maps represent a map of these two complexes. In other words, assume that the rows are exact and the squares commute:

$$\begin{array}{ccccccccc} & & B & \stackrel{b}{\longrightarrow} & C & \stackrel{c}{\longrightarrow} & D & \longrightarrow & 0 \\ & & \downarrow \beta & & \downarrow \gamma & & \downarrow \delta & & \\ 0 & \longrightarrow & B' & \stackrel{b'}{\longrightarrow} & C' & \stackrel{c'}{\longrightarrow} & D'. & & \end{array}$$

Then

$$\ker \beta \to \ker \gamma \to \ker \delta \to \operatorname{coker} \beta \to \operatorname{coker} \gamma \to \operatorname{coker} \delta$$

is exact, where the first two maps are the restrictions of b and c, respectively, the last two maps are the natural maps induced by b' and c', respectively, and the middle map is the so-called **connecting homomorphism**. Furthermore, if b is injective, so is $\ker \beta \to \ker \gamma$; and if c' is surjective, so is $\operatorname{coker} \gamma \to \operatorname{coker} \delta$.

Theorem 2 (Short exact sequence of complexes yields a long exact sequence on homology) *Let* $0 \to C_\bullet{}' \xrightarrow{f_\bullet} C \xrightarrow{g_\bullet} C'' \to 0$ *be a short exact sequence of complexes. Then we have a long exact sequence on homology:*

$$\cdots \to H_{n+1}(C_\bullet{}'') \xrightarrow{\Delta_{n+1}} H_n(C_\bullet{}') \xrightarrow{f} H_n(C_\bullet) \xrightarrow{g} H_n(C_\bullet{}'') \xrightarrow{\Delta_n} H_{n-1}(C_\bullet{}') \xrightarrow{f} H_{n-1}$$

where the arrows denoted by f and g are only induced by f and g, and the Δ maps are the connecting homomorphisms.

Proof. By assumption we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 \to & C'_n & \xrightarrow{f_n} & C_n & \xrightarrow{g_n} & C''_n & \to 0 \\
& \downarrow d'_n & & \downarrow d_n & & \downarrow d''_n & \\
0 \to & C'_{n-1} & \xrightarrow{f_{n-1}} & C_{n-1} & \xrightarrow{g_{n-1}} & C''_{n-1} & \to 0.
\end{array}$$

By the Snake Lemma, for all n, the following rows are exact, and the squares commute:

$$\begin{array}{cccccccc}
& \operatorname{coker} d'_n & \xrightarrow{f_n} & \operatorname{coker} d_n & \xrightarrow{g_n} & \operatorname{coker} d''_n & \to 0 \\
& \downarrow d'_n & & \downarrow d_n & & \downarrow d''_n & \\
0 \to & \ker d'_{n-1} & \xrightarrow{f_{n-1}} & \ker d_{n-1} & \xrightarrow{g_{n-1}} & \ker d''_{n-1}. &
\end{array}$$

Another application of the Snake Lemma yields exactly the desired sequence.
□

complex willdo projinj Hom tensorcompl tensorprodcomp Koszul Koszul23 Koszulfunct homTorExt mapcompl fivelemma sesles Koszuldec Koszreg Koszregex HilbSyz

Yet another way of looking at Koszul complexes:

Proposition 3 *Let R be a commutative ring. Let $C_\bullet$ be a complex over R and let $K_\bullet = K_\bullet(x;R)$ be the Koszul complex of $x \in R$. Then we get a short exact sequence of complexes*

$$0 \to C_\bullet \to C_\bullet \otimes K_\bullet \to C_\bullet[-1] \to 0,$$

with maps on the nth level as follows: $C_n \to (C_n \otimes R) \oplus (C_{n-1} \otimes R) \cong C_n \oplus C_{n-1}$ takes a to $(a,0)$, $C_n \oplus C_{n-1} \to (C_\bullet[-1])_n = C_{n-1}$ takes (a,b) to b.
The differential d on $C_\bullet$ also yields the naturally shifted one on $C_\bullet[-1]$, and the differential δ on $C_\bullet \otimes K_\bullet$ is $\delta_n(a,b) = (d_n(a) + (-1)^{n-1}xb, d_{n-1}(b))$.

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In particular,

$$\cdots \to H_n(C_\bullet) \xrightarrow{x} H_n(C_\bullet) \to H_n(C_\bullet \otimes K_\bullet) \to H_{n-1}(C_\bullet) \xrightarrow{x} H_{n-1}(C_\bullet) \to \cdots$$

The long exact sequence in the corollary breaks into short exact sequences:

$$0 \to \frac{H_n(C_\bullet)}{xH_n(C_\bullet)} \to H_n(C_\bullet \otimes K_\bullet) \to \operatorname{ann}_{H_{n-1}(C_\bullet)}(x) \to 0$$

for all n.

Definition 4 *We say that* $x_1, \ldots, x_n \in R$ *is a* **regular sequence** *on a module* M*, or a* **M-regular sequence** *if* $(x_1, \ldots, x_n)M \neq M$ *and if for all* $i = 1, \ldots, n$*,* x_i *is a non-zerodivisor on* $M/(x_1, \ldots, x_{i-1})M$*. We say that* $x_1, \ldots, x_n \in R$ *is a* **regular sequence** *if it is a regular sequence on the* R*-module* R*.*

Definition 4 *We say that $x_1, \ldots, x_n \in R$ is a* **regular sequence** *on a module M, or a* **M-regular sequence** *if $(x_1, \ldots, x_n)M \neq M$ and if for all $i = 1, \ldots, n$, x_i is a non-zerodivisor on $M/(x_1, \ldots, x_{i-1})M$. We say that $x_1, \ldots, x_n \in R$ is a* **regular sequence** *if it is a regular sequence on the R-module R.*

Corollary 5 *Let $x_1, \ldots, x_n$ be a regular sequence on an R-module M. Then*

$$H_i(K_\bullet(x_1, \ldots, x_n; M)) = \begin{cases} 0 & \text{if } i > 0; \\ \frac{M}{(x_1, \ldots, x_n)M} & \text{if } i = 0. \end{cases}$$

In particular, $K_\bullet(x_1, \ldots, x_n; R)$ is a free resolution of $\frac{R}{(x_1, \ldots, x_n)}$.

Proof. Trivial for $n = 1$. For $n > 1$, let $C_\bullet = K_\bullet(x_1, \ldots, x_{n-1}; M)$, $K_\bullet = K_\bullet(x_n; R)$. By the short exact sequences from the previous page and induction on n, $H_i(K_\bullet(x_1, \ldots, x_n; M)) = H_i(C_\bullet \otimes K_\bullet) = 0$ if $i > 1$, $0 \to 0 = \frac{H_1(C_\bullet)}{x_n H_1(C_\bullet)} \to H_1(C_\bullet \otimes K_\bullet) \to \mathrm{ann}_{H_0(C_\bullet)}(x_n) \to 0$ gives $H_1(C_\bullet \otimes K_\bullet) \cong \mathrm{ann}_{H_0(C_\bullet)}(x_n) = \mathrm{ann}_{M/(x_1,\ldots,x_{n-1})M}(x_n) = 0$, and the exactness of $\frac{H_0(C_\bullet)}{x_n H_0(C_\bullet)} \cong H_0(C_\bullet \otimes K_\bullet)$, and induction on n give that $H_0(K_\bullet(x_1, \ldots, x_n; M)) = H_0(C_\bullet \otimes K_\bullet) \cong \frac{H_0(C_\bullet)}{x H_0(C_\bullet)} \cong M/(x_1, \ldots, x_n)M$. □

Exercise: (Depth sensitivity of Koszul complexes) Let R be a commutative ring and M an R-module. Let $x_1, \ldots, x_n \in R$. Assume that $x_1, \ldots, x_l$ is a regular sequence on M for some $l \leq n$. Prove that

$$H_i(K_\bullet(x_1, \ldots, x_n; M)) = 0$$

for $i = n, n-1, \ldots, n-l+1$.

Exercise: Let R be a commutative ring, $x_1, \ldots, x_n \in R$, and M an R-module. Prove that $(x_1, \ldots, x_n)$ annihilates each $H_n(K_\bullet(x_1, \ldots, x_n; M))$.

Exercise: Let $I = (x_1, \ldots, x_n) = (y_1, \ldots, y_m)$ be an ideal contained in the Jacobson radical of a commutative ring R. Let M be a finitely generated R-module. Suppose that $H_i(K_\bullet(x_1, \ldots, x_n; M)) = 0$ for $i = n, n-1, \ldots, n-l+1$. Prove that $H_i(K_\bullet(y_1, \ldots, y_m; M)) = 0$ for $i = m, m-1, \ldots, m-l+1$.

Theorem 6 *Let (R, m) be a Noetherian local ring. Then the following are equivalent:*

(1) $\mathrm{pd}_R(R/m) \leq n$.

(2) $\mathrm{pd}_R(M) \leq n$ *for all finitely generated R-modules M.*

(3) $\mathrm{Tor}_i^R(M, R/m) = 0$ *for all $i > n$ and all finitely generated R-modules M.*

Proof. Trivially (2) implies (1) and (3). Also, (1) implies (3) if we accept that $\mathrm{Tor}_i^R(M, R/m) \cong \mathrm{Tor}_i^R(R/m, M)$.

Now let M be a finitely generated R-module. Let $P_\bullet$ be its minimal free resolution. By minimality, the image of $P_i \longrightarrow P_{i-1}$ for $i \geq 1$ is in mP_{i-1}. Thus all the maps in $P_\bullet \otimes R/m$ are 0, so that $\mathrm{Tor}_i^R(M, R/m) = P_i/mP_i$. If we assume (3), these maps are 0 for $i > n$, and since P_i is finitely generated, it follows by Nakayama's lemma that $P_i = 0$ for $i > n$, whence $\mathrm{pd}_R(M) \leq n$. □

Theorem 7 *Let (R, m) be a Noetherian local ring. Then the following are equivalent:*
(1) $\mathrm{pd}_R(R/m) \leq n$.
(2) $\mathrm{pd}_R(M) \leq n$ *for all finitely generated R-modules M.*
(3) $\mathrm{Tor}_i^R(M, R/m) = 0$ *for all* $i > n$ *and all finitely generated R-modules M.*

Proof. Trivially (2) implies (1) and (3). Also, (1) implies (3) if we accept that $\mathrm{Tor}_i^R(M, R/m) \cong \mathrm{Tor}_i^R(R/m, M)$.
Now let M be a finitely generated R-module. Let $P_\bullet$ be its minimal free resolution. By minimality, the image of $P_i \to P_{i-1}$ for $i \geq 1$ is in mP_{i-1}. Thus all the maps in $P_\bullet \otimes R/m$ are 0, so that $\mathrm{Tor}_i^R(M, R/m) = P_i/mP_i$. If we assume (3), these maps are 0 for $i > n$, and since P_i is finitely generated, it follows by Nakayama's lemma that $P_i = 0$ for $i > n$, whence $\mathrm{pd}_R(M) \leq n$.
□

Hilbert's Syzygy Theorem. Let M be a finitely generated module over the polynomial ring $R = k[X_1, \ldots, X_n]$ over a field k. Then $\mathrm{pd}_R(M) \leq n$.
(The same proof as above shows it for graded finitely generated modules!)