Remark on notation: In the last lecture, a non-traditional way of coordinating partitions was used.

\[ \lambda = (5, 3, 1, 1, 0, 0) \]

For every such partition, we looked at

\[ p_k(\lambda) = \text{regularization of the sum} \sum_{i=1}^{\infty} \left( \lambda_i - i + \frac{1}{2} \right)^k \]

The regularization of \( p_k(\lambda) \) used in the last lecture is

\[ p_k(\lambda) = \sum_{i=1}^{\infty} \left[ \left( \lambda_i - i + \frac{1}{2} \right)^k - (-i + \frac{1}{2})^k \right] + \left( 1 - \frac{1}{2^k} \right) \sum_{-k}^{k} (-1)^k \]

\[ = k! \left[ z^k \sum_{i=1}^{\infty} e^{\frac{z(\lambda_i - i + \frac{1}{2})}{2}} \right] \]
Remark: Given a power series \( f(\varepsilon_1, \ldots, \varepsilon_n) \) in variables \( \varepsilon_1, \ldots, \varepsilon_n \), we denote by
\[
[\varepsilon_{k_1} \cdots \varepsilon_{k_n}] f(\varepsilon_1, \ldots, \varepsilon_n)
\]
the coefficient of \( f \) in front of the monomial \( \varepsilon_{k_1} \cdots \varepsilon_{k_n} \).

In the notation we just introduced, we can rewrite the last formula from Rahul's talk as
\[
(\star) \quad \left< \prod_{i=1}^{n} T_{k_i} (\rho) \right>_{d} = [\varepsilon_{k_1} \cdots \varepsilon_{k_n}] \left( \sum_{|\lambda|=d} \left( \frac{\text{dim} \lambda}{d!} \right)^2 \prod_{i=1}^{n} \epsilon(\lambda, \varepsilon_i) \right)
\]
where
\[
\epsilon(\lambda, \varepsilon_i) := \sum_{i=1}^{\infty} \varepsilon^{(\lambda_i - i^2 + \frac{1}{2})}
\]

Remark: If the target for our GW theory is a curve \( X \) of genus \( g \), we need only replace the exponent 2
in the formula (\( \star \)) with \( \mathcal{L} \cdot \mathcal{H} \).

The formula (\( \star \)) can be understood even better in terms of infinite wedge representations.

Infinite wedge representations:

Let \( V = \bigotimes_{K \in \mathbb{Z}}^{+1} \mathbf{C} \cdot K \)

Then \( \Lambda^{\infty} V \) is the collection of all \( V_\lambda \):

\[
\Lambda^{\infty} V \ni V_\lambda = (\lambda_1 - \frac{1}{2}) \Lambda \lambda_2 - \frac{3}{2} \Lambda \lambda_3 - \frac{5}{2} \Lambda \ldots \\
V_\varnothing = \frac{i}{2} \Lambda \frac{3}{2} \Lambda \frac{-5}{2} \Lambda \ldots
\]

\( \mathfrak{gl}(V) = \mathfrak{gl}(\infty) \) "acts" on \( \Lambda^{\infty} V \) via

\[
\mathfrak{gl} \in \mathfrak{gl}(V) \quad \text{for} \quad \mathfrak{gl} = \begin{pmatrix} 0 \\ \vdots \\ i-\mathfrak{gl} \end{pmatrix}
\]

\[
\Lambda^{\infty + 1} V_\lambda = \sum_{\mu = \lambda + \Delta} V_\mu
\]
In fact one checks that
\[ [x_k, x_\ell] = k \delta_{k+\ell} \]
and so what we get is that a central extension of \( gl(\infty) \) acts. Moreover one checks that
\[ (x_{-1})^d V_\emptyset = \sum_{|\lambda| = d} (\text{dim } \lambda) V_\lambda \]

We can also consider \( \xi(z) \in gl(V) \)
s.t.
\[ \xi(z) = \begin{pmatrix} e^{\frac{3\pi}{4}} & 0 \\ 0 & e^{\frac{\pi}{4}} \end{pmatrix} \]
\[ \text{i.e. } \xi(z) \xi^{-1} = e^{\frac{\pi}{2} \overline{z}} \]

From this formula we get
\[ \xi(z) V_\lambda = e(z, z) V_\lambda \]
and so the formula (*) can be rewritten as
\[ \left< \prod_{i=1}^{n} T_{e_{z_i}}(p) \right>_{d_1} = \left[ \overline{e_1} \cdots \overline{e_n} \right] \]
\[ = \left( \frac{(d_1)!}{d_1!} \prod_{i=1}^{n} \xi(z_i) \left( \frac{(d_{-1})_i}{d_i} \right)^d V_\emptyset \right) V_\emptyset \]
where \( (\cdot, \cdot) \) is the inner product on \( \Lambda^2 V \) for which all \( V_k \) are orthonormal.

**Notation:** Given \( A \in \text{End}(\Lambda^2 V) \) define

\[
\langle A \rangle = (A \psi, \psi)
\]

Thus we get that \((\ast)\) can be rewritten as

\[
\left\langle \prod_{i=1}^{n} T_{k_i}(\psi) \right\rangle^0_{\mathbb{P}^1} = [z_1^{k_1+1} \cdots z_n^{k_n+1}] \left\langle \frac{(d_1)^d}{d!} \prod \frac{d(z_i)}{d!} \right\rangle
\]

**Relative Theory**

\[ \mu = (2, 1, 1) \]

\[ V = (3, 1) \]

\[ |\mu| = |V| = d \]
Now

\[ \langle \mu | \prod_{k=0}^n T_k(p) | \nu \rangle_d \]

\[ \frac{1}{\text{Aut } \mu \text{ Aut } \nu} \prod_{j=1}^m \mathcal{E}(z_j) \prod \mathcal{E}(z_j - \nu_j) \]

\[ \mu = (a_1, \ldots, a_d) \quad \nu = (1, \ldots, 1) \]

\[ d \text{ times} \quad d \text{ times} \]

Example: \( n = 1 \). We need to compute \( \mathcal{O} \leq \text{connected } \mathfrak{g} \mathfrak{w} \) invariants

\[ \langle \prod_{j=1}^m \mathcal{E}(z_j) \prod \mathcal{E}(z_j - \nu_j) \rangle = \]

\[ = \sum \mathbb{Z}^{10+1} \langle \mu | T_k(p) | \nu \rangle^0 \]

(Remark: \( \langle \mu | T_k(p) | 1 \rangle^0 = \Gamma_1^0 (\mu, (k+1)) \))

In the notation from Rahul's talk.)

In order to carry out this computation, note first that

\[ \Delta \nu \phi = 0 \quad \text{for } k > 0. \]

Now extend the definition of \( \mathcal{E} (z) \) to
\[ \xi_0(\xi) \in \text{End} (\Delta V) ; \]

\[ \xi_0(\xi) \triangleq e^{2\xi} i - e \]

These operators give \( \xi_0(\xi) = \xi(\xi) \)
and have commutation relations

\[ [L_\xi, \xi(\xi)] = (1 - e^{-2\xi}) \xi(\xi). \]

Then we can commute \( dp_i \) through \( \xi(\xi) \) to get

\[ \sum_{\xi} e^{2\xi} \langle m | T_\xi(p) | V \rangle = \]

\[ = \frac{1}{\text{Aut} m \text{ Aut} V \prod p_i \prod v_i} \cdot \prod \frac{P_{\xi p_i}}{P_{\xi v_i}} \cdot \frac{\prod \tilde{f}(p_{\xi i}) \tilde{f}(\xi v_{\xi i})}{\tilde{f}(\xi)} \]

where \( \tilde{f}(\xi) = \frac{\sinh (\frac{\xi}{2})}{(\frac{\xi}{2})} \)

If we have an elliptic curve \( E \) as a target, then

\[ \sum_{\nu_1, \ldots, \nu_d} q^{d} \langle \prod T_{\xi_i}(p) \rangle d^i \varepsilon_{\xi_1, \ldots, \xi_d, \nu_1, \ldots, \nu_d} \]

charge 0

subspace
Here
\[ H_i = i \]
and so \[ H \nu_\lambda = 1_{\lambda!} \nu_\lambda. \]

**Remark:** The RHS was computed by Bloch-Okounkov in terms of determinants of \( \Theta \)-functions.

---

**Equivalent GW theory of \( \mathbb{P}^1 \)**

\[ 0 \to \infty \]
\[ T = \mathbb{C}^x \]

\[ [03], [20] \in H^0_T(\mathbb{P}^1) \]

In fact \( H^0_T(\mathbb{P}^1) \) is a module over \( H^0_T(\text{pt}) = \mathbb{C}[t] \) and the identity class

\[ 4t = \frac{[03] - [20]}{t} \in H^0_T(\mathbb{P}^1) \]

Now

\[ \langle \ldots \rangle = \sum_{d, g} u^{2g-2} g^d \langle \ldots \rangle_{g, d} \]

and the corresponding generating function is

\[ \mathcal{F}(z_0, z_1, \ldots; w_0, w_1, \ldots) = \langle \exp \left( \sum_{i=0}^{\infty} \ell_i z^{i} + \ell_i (\infty) w^{i} \right) \rangle^0 \]
It turns out that

\[
F(z_0, \ldots, z_n) = \log \left< e^{ \sum Z_k A_{k+1} - d_1 \left( \frac{1}{u_2} \right)^n} \right>
\]

for some explicit

\[
A_k = \left( \begin{array}{c} \vdots \end{array} \right), \quad A_k^* := A_k^T |_{k \to 0}
\]

In fact, already the existence of a formula like that (regardless of the explicit form of the $A_k$'s) implies

\[\frac{\partial^2}{\partial z_0 \partial z_n} F = e^\Delta F\]

where

\[\Delta F = F |_{s \to s+u} + F |_{s \to s-u} - 2F\]

and

\[\frac{\partial}{\partial s} = \text{conj} \quad \text{to} \quad \frac{\partial}{\partial t} = \frac{[0] - [\omega]}{t}\]

i.e.

\[\frac{\partial}{\partial s} F = \left< \tau_0 (A) \exp(-\omega) \right>\]
Remark: \( \frac{\partial x}{\partial s} = \frac{1}{t} \left( \frac{\partial x_0}{\partial s} - \frac{\partial x_0}{\partial s_0} \right) \).

Remark: The above Toda equation corresponds to the lowest order Plücker identity in the semi-infinite matrix \( F \):

\[
\begin{vmatrix}
\det & \det \\
& \\
\end{vmatrix} - \begin{vmatrix}
\det & \det \\
\det & \\
\end{vmatrix} = \begin{vmatrix}
\det & \\
\det & \\
\end{vmatrix}
\]

(One has to rewrite this identity for 1st and last rows of a matrix rather than 1st + last, so that it will make sense for infinite matrices.)

All the other Plücker identities give rise to higher equations in the Toda hierarchy.
To write down $A_k$ explicitly we look at a generating function for the $A_k$'s:

$$A(z) = \sum_{k \in \mathbb{Z}} z^k A_k = \frac{1}{4} \int (u z)^{kz} \psi_k(u z) \frac{1}{(1 + u z) \ldots (1 + k z)}.$$

**Note:** In terms of the $A_k$ we can write the $n$-point function as

$$\langle \prod \exp \left( \frac{\mathbf{d}_1}{u z} \right) \exp \left( \frac{-\mathbf{d}_1}{u z} \right) \prod A_k^\ast (w_z) \rangle =$$

- multiple hypergeometric sum