RANDERS METRICS OF POSITIVE CONSTANT CURVATURE

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1. Introduction

(a) Some Results on Finsler (Randers) Metrics of Constant Curvature

- H. Akbar-Zadeh (1988), under some growth constraints on Cartan tensor, proved that Finsler manifolds with $K = 0$ are locally Minkowski and with $K < 0$ are Riemannian.
- R. Bryant (1996-1998) constructed projectively flat Finsler metrics on S^n with $K = 1$.
- H.Yasuda and H. Shimada (1977) found sufficient conditions for a Randers metric to be of constant curvature.
- D. Bao and C. Robles (2001) found necessary and sufficient conditions for a Randers metric to be of constant curvature.
- D. Bao and Z. Shen (2000) constructed a family of non-projectively flat Randers metrics on S^3 with $K > 1$.
- Z. Shen (2001) constructed on S^2 Randers metrics with $K = 1$ (the 1-form b vanishes at two points).

(b) The Purpose of our Talk

- To construct on S^{2m+1}, $m \geq 1$, a family of non-projectively flat Randers metrics with $K > 0$.

- To classify non-singular Randers manifolds with $K > 0$, provided they satisfy Bao-Robles condition.

2. Sasakian Space Forms and Randers Metrics

Let M be a $(2m+1)$-dimensional manifold endowed with:

- a tensor field ϕ of type (1, 1)
- a vector field ξ
- a 1-form η
- a Riemannian metric a.

If the following are satisfied:

$$\varphi^2 = -I + \eta \otimes \xi; \eta(\xi) = 1;$$

$$a(\varphi X, \varphi Y) = a(X, Y) - \eta(X)\eta(Y),$$

then M is called *almost contact metric manifold.*

When we have

$$(\nabla_X \varphi) Y = a(X, Y)\xi - \eta(Y)X,$$

we say that M is a *Sasakian manifold.* A *Sasakian space form* is a Sasakian manifold of constant φ-sectional curvature $K(X, \varphi X) = c$, and it is denoted by $M(c)$.

The curvature tensor of $M(c)$ has a special form:

$$
\begin{aligned}
(1) \qquad R_{hijk} &= \frac{c+3}{4}\{a_{jh}a_{ik} - a_{kh}a_{ij}\} \\
&+ \frac{1-c}{4}\{\eta_j\eta_h a_{ki} + \eta_k\eta_i a_{jh} - \eta_k\eta_h a_{ij} \\
&\quad -\eta_{h|j}\eta_{i|k} + \eta_{h|k}\eta_{i|j} + 2\eta_{h|i}\eta_{k|j}\}
\end{aligned}
$$

Typical examples of $M(c)$:

1. S^{2m+1} bears a family of structures with $c > -3$.

2. $\mathbb{R}^{2m+1}$ is an example for $c = -3$.

3. $\mathbb{B}^m \times \mathbb{R}$, where $\mathbb{B}^m$ is a bounded domain in $\mathbb{C}^m$ is an example for $c < -3$.

Non-singular Randers Metrics

Let M be n-dimensional manifold endowed with a Riemannian metric $a = a_{ij}(x)$ and a non-zero 1-from $b = (b_i(x))$ with $||b|| < 1$. Then

$$\mathbb{F}^n = (M, F), \quad \text{where} \quad F(x, y) = \sqrt{a_{ij}(x)y^i y^j} + b_i(x)y^i$$

is called a *non-singular Randers manifold.*

The *flag curvature* of $\mathbb{F}^n$ is (cf. Bao-Chern-Shen)

$$K(\ell, V) = \frac{V^i R_{ij} V^j}{g_{ij}V^i V^j - (g_{ij}\ell^i V^j)^2}.$$

$\mathbb{F}^n$ is of constant curvature $K \Longleftrightarrow R_{ij} = K h_{ij}, \;\; h_{ij} = g_{ij} - \ell_i \ell_j$ (angular metric).

3. Non-Projectively Flat Randers Metrics of Positive Constant Curvature on $S^{2m+1}, m \geq 1$.

Bao and Shen constructed non-projectively flat Randers metrics of positive constant curvature on S^3.

By using an interesting interrelation between Sasakian space forms and Randers manifold we extend the result to $S^{2m+1}, m \geq 1$.

First, from Yasuda-Shimada Theorem we have the following:

THEOREM $(Y-S)$. *Let $\mathbb{F}^n = (M, F, a_{ij}, b_i)$ be a Randers manifold such that:*

(i) $\|b\|$ *is a constant;*

(ii) $b_{i|j} + b_{j|i} = 0$

(iii) The curvature tensor of the Riemannian manifold $(M, a_{ij}(x))$ is given by

$$(2) \quad \begin{aligned} R_{hijk} &= K(1-\|b\|^2)\{a_{jh}a_{ik} - a_{hk}a_{ij}\} \\ &+ K\{b_i b_k a_{hj} + b_h b_j a_{ik} - b_i b_j a_{hk} - b_h b_k a_{ij}\} \\ &- b_{h|j}b_{i|k} + b_{h|k}b_{i|j} + 2b_{h|i}b_{k|j}, \end{aligned}$$

where K is a positive constant.

Then $\mathbb{F}^n$ is a Randers manifold of constant curvature K.

Remark the similitude between the formulas (1) and (2).

Let $M(c)$ be a Sasakian space form with Sasakian structure (φ, ξ, η, a) and $c \in (-3, 1)$. Consider $\alpha = \frac{1}{2}\sqrt{1-c}$ and define $b = \alpha\eta$. Then it follows $0 < ||b|| < 1$, so

$$F(x, y) = \sqrt{a_{ij}(x)y^i y^j} + b_i(x)y^i$$

is a non-singular Randers metric on $M(c)$.

Then we prove:

LEMMA 1. *$\mathbb{F}^{2m+1}(M(c), F, a_{ij}, b_i)$ is a Randers manifold of constant flag curvature $K = 1$.*

LEMMA 2. *Let $\mathbb{F}^n = (M, F)$ be a Randers manifold of constant curvature $K = 1$. Then for any constant $K^* > 0$ there exists a Randers metric F^* such that $F^{n*} = (M, F^*)$ is a Randers manifold of curvature K^*.*

THEOREM (B-F). *Let $M(c)$ be a Sasakian space form such that $c \in (-3, 1)$. Then for any $K > 0$ there exists a non-singular Randers metric of constant curvature K that is not projectively flat.*

COROLLARY *Let S^{2m+1} as a Sasakian space form with $c = -3 + \frac{4}{\varepsilon}, \varepsilon > 1$. Then for any $K > 0$ there exists a non-singular Randers metric of constant curvature K that is not projectively flat.*

4. On the Classification of Randers Manifolds of Positive Constant Curvature

Let $\mathbb{F}^n = (M, F, a_{ij}, b_i)$ be a non-singular Randers manifold of positive constant curvature K, satisfying Bao-Robles condition:

$$(*) \qquad curl_{b_j} = b^i \left(b_{i|j} - b_{j|i}\right) = 0.$$

Then we proved the following classification theorem.

THEOREM (B-F). If $\mathbb{F}^n = (M, F, a_{ij}, b_i)$ is as above, and M is simply connected and complete manifold, then M is diffeomeorphic to the unit sphere S^{2m+1}.

Main Steps in the Proof:

Due to $(*)$, the Yasuda-Shimada conditions are also necessary conditions. For $K = 1$ we define

$$\eta = \frac{1}{||b||} b, \ \xi = \eta^\sharp, \varphi X = -\nabla_X \xi.$$

Then we prove that M is a Sasakian space form with respect to (φ, ξ, η, a), and

$$c = 1 - 4||b||^2 \in (-3, 1).$$

Finally, we apply Tanno's Theorem:

Let $M(c)$ be a $(2m+1)$-dimensional simply connected and complete Sasakian space form with $c > -3$. Then $M(c)$ is "isomorphic" to S^{2m+1}.

5. Conclusions

According to our results, the classification of Randers manifolds of positive constant curvature will be complete when we classify:

(a) Non-singular $\mathbb{F}^n$ without Bao-Robles condition.

(b) Singular $\mathbb{F}^n$.

Remark. Both classes (a) and (b) are not empty. Indeed, Shen's example with $K = 1$ on S^2 and Bao-Robles example with $K = 1$ on S^3, fall in (b). Also, from these examples we may conclude that there exist Randers metrics with $K = 1$, which fall in (a), but they are defined on open sets of $\mathbb{R}^2$ or $\mathbb{R}^3$.

Conjecture. If $\mathbb{F}^n = (M, F, a_{ij}, b_i)$ falls in (a), then M must be an open set of $\mathbb{R}^n$.

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