\[ S = k[x_0, \ldots, x_r] = \text{Sym}_W W \]
\[ E = \Lambda(W^*) \quad \text{exterior algebra} = \Lambda^e \left( x_0, \ldots, x_r \right) \]
\[ \text{Prop} \quad \text{Linear free } S\text{-complexes} \cong \text{graded } E\text{-modules} \]
\[ \text{Graded } S\text{-modules} \cong \text{linear free } E\text{-complexes} \]

Sand E are Koszul dual algebras.

\[ x_i \text{'s have deg } 1, \text{ so functionals in } W^* \text{ have deg } -1 \]

\[ P \text{ a graded } E\text{-module } P = \oplus P_i \]

\[ \cdots \rightarrow S^1 \otimes P_i \rightarrow S^1 \otimes P_{i-1} \rightarrow S^1 \otimes P_{i-2} \rightarrow \cdots \]

\[ \sum_{j} x_j \otimes \varepsilon_j p \rightarrow \sum_{j} x_j y_j \otimes \varepsilon_j p \]

So this is a differential, using commutativity in S & associativity & anticommutativity in E.
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$M = \bigoplus M_i$, an $S$-module.$\quad \Rightarrow \quad \text{IR}(M) \rightarrow E \otimes M_i \rightarrow E \otimes M_i \rightarrow \cdots$

write $P = \text{Hom}_k(P, k)$; say $P$ a f.g. graded $E$-module.

Prop 1) $L = L(P)$ is a subcomplex of the linear strand of a minimal free resolution.

$\iff \hat{P}$ is finitely generated in degree 0,

(by $\hat{P}_0$).

$S \otimes \hat{P}_i \rightarrow \cdots \rightarrow S \otimes \hat{P}_0$

2) $L = L(P)$ is the linear strand of a minimal free res

$\iff \hat{P}$ is linearly presented.

$E_1 \rightarrow E_0 \rightarrow \hat{P}$

3) $L = L(P)$ is a free resolution $\iff \hat{P}$ has a linear free res.

(*) $\text{Hi}(L(P))_{tt} = \text{Tor}_d^E(\hat{P}, k)_{t-d}

The case $P = E$ is just the Koszul complex.

$\text{IR}(S) = \quad E \rightarrow E \otimes W \rightarrow E \otimes \text{Sym}_2 W \rightarrow \cdots$

is the injective resolution of $\text{Kas} \, E$-module.

this is the Cartan resolution.

cf. Cartan-Eilenberg

in general, for any graded $E$-module $M$,

$\text{IR}(M)$ is exact starting at $\text{reg} \, M$

the Castelnuovo-Mumford regularity of $M$. 
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Given a finitely generated graded S-module M, what's the length of the linear strand of the minimal resolution of M? How long could it be?

**Example 1.** If M has 1 generator \( m \),
then the length is \( \dim \text{soc} Wm = 0^3 = A_0 \).

**Example 2.**
\[
S^l(-1) \rightarrow S^2 \rightarrow N \rightarrow 0
\]
\[
S = NS^l(1) \oplus \text{Sym}_0 S^2 \rightarrow \cdots \rightarrow NS^{l-2} \oplus \text{Sym}_{l-2} S^2
\]
\[
S^{l-1}(1) \oplus \text{Sym}_{l-1} S^2 \rightarrow \text{Sym}_1 S^2 \rightarrow M = \text{Sym}_l N \rightarrow 0
\]
has resolution of length \( l \).

Check: case \( l = 2 \): \( 0 \rightarrow NS^{2-1} \rightarrow S^{l-1} \oplus S^{l-2} \rightarrow \text{Sym}_2 S^2 \)

No element of M is annihilated by a linear form.
M has \( l+1 \) generators.

Let M be graded, \( M = M_0 \oplus M_1 \oplus \cdots \), \( M_0 \neq 0 \).

\( A(M) = \{ x, \langle m \rangle \in W \times \mathcal{P}(M_0) \text{ s.t. } x m = 0^3 \} \).

In Example 1: \( \text{ann}_W (m) \neq \langle m \rangle \)
with dimension = dim annihilator.

In Example 2: \( 0 \times \mathcal{P}(M_0) \) has dim 1.

**Theorem (Mark Green)**
\( \dim A(M) \geq \) length of linear strand of resolution of M.
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other stronger conjectures are in
Eisenbud & Koh, 1991

suppose have linear \( S \otimes P_i \rightarrow S \otimes P_i \)
\( P_i \rightarrow W \otimes P_{i-1} \leftarrow V \otimes P_i \rightarrow P_{i-1} \)

Let \( L = \text{linear strand} = \Omega L(e) \)
\( \mathcal{P} \) is linearly presented \( E^b(l) \rightarrow E^a \rightarrow \mathcal{P} \rightarrow \mathcal{O} \)
\( L \) has length \( < \dim \mathcal{A}(M) \)
\( \iff (V) \dim \mathcal{A}(M) + 1 \mathcal{P} = \mathcal{O} \)

annihilates - must show

(1) find elements that annihilate a module

(2) show you found a lot

- try \( \mathcal{P} \) is annihilated by the "exterior minors"

(Fitting ideal) of \( \mathcal{P} \)

Case \( b = 1 \) local presentation:
\[
\begin{align*}
E(1) & \rightarrow E^a \rightarrow \mathcal{P} \rightarrow \mathcal{O} \\
\epsilon_i & \in \mathcal{P} \quad \left( \begin{array}{c} \epsilon_i \\ \epsilon_a \end{array} \right) \\
\sum \epsilon_i p_i = 0 \\
\text{The } \epsilon_i \text{ annihilates: permanent of mtx obtained by repeating } (\epsilon_a) \text{ sufficiently.}
\end{align*}
\]

\[
(l_1 \cdots l_k) p_i = \sum_{j \leq l} \epsilon_i p_i = 2 (T l e_j) \sum \epsilon_j p_j = 0.
\]