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On the number of connected components of smooth real varieties

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(in RAAG, de Gruyter, 1995, with Becker)

Theorem : Let Y be a smooth projective irreducible variety over $\mathbb{R}$, with function field $K = \mathbb{R}(Y)$, let $\mid \pi_0(Y(\mathbb{R})) \mid$ be the the number of connected components of $Y(\mathbb{R}) \neq \emptyset$ then :

$$\mid \pi_0(Y(\mathbb{R})) \mid = 1 + \log_2[(K^{*2} \cap \sum K^4) : (\sum K^{*2})^2]$$

$(\sum K^{*2})^2$ is a subgroup of $(K^{*2} \cap \sum K^4)$ because any square of a sum of squares is a sum of fourth powers, and the index of the subgroup $(\sum K^{*2})^2$ in the group $(K^{*2} \cap \sum K^4)$ is a power of 2.

The proof uses the connected components of $M(K)$ the space of $\mathbb{R}-places$ since we can prove that

$$\mid \pi_0(Y(\mathbb{R})) \mid = \mid \pi_0(M(\mathbb{R}(Y))) \mid$$

and that for any real field K :

$$\mid \pi_0(M(K)) \mid = 1 + \log_2[K^{*2} \cap \sum K^4 : (\sum K^{*2})^2]$$

K formally real field ($-1 \notin \sum K^2$), P order of K : $P + P \subseteq P$, $P.P \subset P$, $-1 \notin P$, $P \cup -P = K$

$$A(P) = \{a \in K \mid \exists r \in \mathbb{Q}, -r \leq_P a \leq_P r\}$$

is a valuation ring of maximal ideal

$$I(P) = \{a \in K \mid \forall r \in \mathbb{Q}, -r \leq_P a \leq_P r\}$$

The residue field is $k_v = A(P)/I(P)$.

P induces over k_v an archimedean order $\overline{P}$

$(k_v, \overline{P})$ can be embedded uniquely in $(\mathbb{R}, \mathbb{R}^2)$

The $\mathbb{R} - place$ associated to P is $\lambda_P : K \longrightarrow \mathbb{R} \cup \{\infty\}$

defined by the following commutative diagram :

$$K \xrightarrow{\lambda_P} \mathbb{R} \cup \{\infty\}$$

$$\pi \searrow k_{v \cup \{\infty\}} \nearrow i$$

Explicitely : $\lambda_P(a) = \infty$ if $a \notin A(P)$

and if $a \in A(P)$, $\lambda_P(a) = \inf\{r \in \mathbb{Q} \mid a \leq_P r\}$

$$= \sup\{r \in \mathbb{Q} \mid r \leq_P a\}$$

The set of $\mathbb{R}-places$ is:

$$M(K) = \{\lambda_P \mid P \in \chi(K)\}$$

($\chi(K)$ denotes the space of orders of the field K).

The space $M(K)$ is given the coarsest topology such that the evaluation applications defined $\forall a \in K$ below are continuous

$$e_a : M(K) \longrightarrow \mathbb{R} \cup \{\infty\}$$

$$\lambda \mapsto \lambda(a)$$

Now the following map Λ is continuous, closed and surjective :

$$\Lambda : \chi_K \longrightarrow M(K)$$

$$P \mapsto \lambda_P$$

$M(K)$ is a compact Hausdorff space.

The topology of $M(K)$ is also the quotient topology inherited from χ_K equipped with the Harrison topology generated by the open-closed sets

$$H(a) = \{P \in \chi_K \mid a \in P\}.$$

χ_K is a totally disconnected compact space.

The real holomorphy ring of a formally real field K is defined as

$$H(K) = \cap A(P)$$

where P ranges over $\chi(K)$

and $A(P) = \{a \in K \mid \exists r \in \mathbb{Q} - r \leqslant_P a \leqslant_P r\}$.

The real holomorphy ring is also equal to the intersection of all real valuation rings of K .

Real spectrum of $H(K)$:

$Sper H(K) = \{\alpha = (p,\overline{\alpha}) \mid p \in spec H(K), \overline{\alpha}$ *order of* $quot(H(K)/p)\}$

Theorem : the following diagram is commutative

$$\begin{array}{ccc} \chi(K) & \xrightarrow{speri} & MinSperH(K) \\ \downarrow \Lambda & & \downarrow sp \\ M(K) \overset{res}{\rightarrow} & Hom(H(K),\mathbb{R}) & \overset{j}{\rightarrow} MaxSperH(K) \end{array}$$

The horizontal mappings are homeomorphisms, and the vertical one are continuous surjections .

$speri$ is defined by $P \mapsto P \cap H(K)$,

sp by $\alpha \mapsto \alpha^{\max}$,

res by $\lambda \mapsto \lambda_{|H(K)}$

j by $\varphi \mapsto \varphi^{-1}(\mathbb{R}^2)$.

The spaces are compact and $M(K)$ and $MaxSperH(K)$ have quotient topologies inherited from Λ and sp.

Sketch of proof of

$$|\pi_0(Y(\mathbb{R}))| = |\pi_0(M(\mathbb{R}(Y)))|$$

We use the center map $c : M(K) \to Y(\mathbb{R})$, defined by $c(\lambda) = c(V_\lambda)$ the unique point x (Y projective) whose local ring $\mathcal{O}_x$ is dominated by V_λ the valuation ring associated to the $\mathbb{R}-$place.

- In that case c is surjective, the central points being the closure of the regular points. And one can prove that c is continuous.

- Bröcker proved that the fiber of a central point has a finite number of connected components, and if x is regular then the fiber is connected.

Lemma: if an application, from a compact space X to another compact space Y, is surjective and continuous, and if each fiber is connected, then it induces a bijection beetween $\pi_0(X)$ and $\pi_0(Y)$.

Idea of proof for :

$$| \pi_0(M(K)) | = 1 + \log_2[K^{*2} \cap \sum K^4 : (\sum K^{*2})^2]$$

This comes from the separation of connected components of $M(K)$ using elements $\beta \notin \sum K^2$ such that $\beta^2 \in \sum K^4$.

For the proof we use the units of the real holomorphy ring and prove :

$$| \pi_0(M(K)) | = \log_2[E : E^+]$$

where E are the units of the real holomorphy ring $H(K)$ and where $E^+ = E \cap \sum K^2$

And we prove that $K^{*2} \cap \sum K^4 / (\sum K^{*2})^2$ is isomorphic to $E/(E^+ \cup -E^+)$.

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Definition : Let K *be a commutative formally real field,* $P \subset K$ *is an ordering of exact level* n *if :* $\sum K^{2n} \subset P$, $P + P \subset P$, $P.P \subset P$ *(hence* P^* *is a subgroup of* K^**) and we have* $K^*/P^* \simeq \mathbb{Z}/2n\mathbb{Z}$.

Level 1 orderings are the total usual orders.

Theorem : $\sum K^{2n} = \bigcap_{\text{level of } P \text{ divides } n} P$

Theorem : Let p be a prime, $\sum K^2 \neq \sum K^{2p} \Longleftrightarrow K$ *admits orderings of level* p.

Another look : higher level signatures

Definition : a signature of level n is a morphism of abelian groups

$$\sigma : K^* \longrightarrow \mu_{2n}$$

such that the kernel is additively closed.

Remark : then $P = \ker \sigma \cup \{0\}$ is an ordering of higher level, and its level divides n.

Example : $K = \mathbb{R}((X))$ the two usual orders are

$$P_+ = K^2 \cup XK^2 \text{ and } P_- = K^2 \cup -XK^2$$

and for any prime p there exist two orderings of level p :

$$P_{p,+} = K^{2p} \cup X^p K^{2p} \text{ and } P_{p,-} = K^{2p} \cup -X^p K^{2p}$$

Theorem (Becker) : the following are equivalent :

(1) Λ *is a bijection ;*

(2) $\forall a \in K \quad a^2 \in \sum K^4$;

(3) every real valuation of K *has a 2-divisible value group ;*

(4) K *does not admit any ordering of exact level* 2 .

(1) in the theorem follows from :

Theorem : $\lambda_P = \lambda_Q \Leftrightarrow P$ *and* Q *belong to a 2-primary chain of orderings (2-power levels)*

Definition (Harman) : a 2-primary chain of orderings is $(P_n) = (P_0, P_1, ..., P_n, ...)$, P_0 *being a usual order,* P_n *an ordering of level* 2^{n-1} *such that*

$$P_n \cup -P_n = (P_0 \cap P_{n-1}) \cup -(P_0 \cap P_{n-1})$$

Note that all the P_n induce the same archimedean order $\overline{P_n}$ on the residue field k_v of the valuation v associated to the Becker valuation ring $A(P) = \{a \in K \mid \exists r \in \mathbb{Q} \; -r \leqslant_P a \leqslant_P r\}$

Theorem : λ_P *and* λ_Q *are in two distinct connected components of* $M(K)$ *if and only if :*

$$\exists \beta \in K^* \; (\beta \in P \cap -Q \; et \; \beta^2 \in \sum K^4)$$

If exists b, such that $b \notin \sum K^2$ and $b^2 \in \sum K^4$, then does not exist $P \in H(b)$ and $Q \in H(-b)$ such that $\lambda_P = \lambda_Q$.

Otherwise $b \notin (P \cap Q) \cup -(P \cap Q)$ and $\lambda_P = \lambda_Q$ imply that there exists an ordering of level 2, P_2, such that

$$P_2 \cup -P_2 = (P \cap Q) \cup -(P \cap Q)$$

with $b^2 \notin P_2$ hence $b^2 \notin \sum K^4 = \cap P_{2,i}$, because $b^2 \notin P_2 \Leftrightarrow b \notin P_2 \cup -P_2$.

$\Leftarrow$

Suppose that λ_P and λ_Q are in the same connected component C of $M(K)$ $(P \neq Q)$,

and that there exists $b \in P \cap -Q$ with $b^2 \in \sum K^4$,

Λ being closed $C \cap \Lambda(H(b))$ and $C \cap \Lambda(H(-b))$ are a partition of C into two non empty closed sets, impossible.

$\Rightarrow$

If λ_P and λ_Q are in C and C' two distinct connected component, $M(K)$ being compact and Hausdorff, there exists an open-closed set $U \supset C$ and $U^c = M(K)\backslash U \supset C'$.

Let $X = \Lambda^{-1}(U)$ and $Y = \Lambda^{-1}(U^c)$, X and Y form a partition of $\chi(K)$; Λ being surjective

$$\Lambda^{-1}(\Lambda(\Lambda^{-1}(U))) = \Lambda^{-1}(U)$$

hence $\Lambda^{-1}(\Lambda(X)) = X$ and also $\Lambda^{-1}(\Lambda(Y)) = Y$.

Harman's following lemma shows that exits b such that $X = H(b)$ and $Y = H(-b)$ with $b^2 \in \sum K^4$, then we have $b \in P \cap -Q$ and $b^2 \in \sum K^4$.

Harman (Co. Math. 8 (1982)) *If* $\chi(K) = \chi_1 \cup \chi_2$ *where* χ_1 *and* χ_2 *are open-closed disjoint sets such that* $\Lambda^{-1}(\Lambda(\chi_1)) = \chi_1$ *and* $\Lambda^{-1}(\Lambda(\chi_2)) = \chi_2$, *then there exists* a *such that* $\chi_1 = H(a)$ *and* $\chi_2 = H(-a)$ and $a^2 \in \sum K^4$.

Definition (B. Jacob 1981) : a valuation fan is a preordering T such that there exists a real valuation v, compatible with T, (means $1 + I_v \subset T$), inducing an archimedean ordering on the residue field k_v.

Example : higher level orderings are valuation fans.

Definition (N. Schwartz 1990) : a generalized signature is a morphism of abelian groups

$$\sigma : K^* \to G$$

such that the kernel is a valuation fan

There exist many notions of real closure under algebraic extensions of either higher level orderings or signatures, chains of signatures, valuation fans, generalized signatures, chains of valuation fans.

All these can be unified in one theory : Henselian Residually Real-Closed Fields (HRRC fields)

Henselian Residually Real-Closed Fields (HRRC)

v henselian valuation , k_v real-closed field

closed for generalized signature or for valuation fan

HRRC field of type S ($p \notin S \Rightarrow \Gamma_v$ p-divisible)

S–generalized real-closed field	Rolle field
$p \notin S$ $\Gamma_v \Rightarrow p$-divisible	Γ_v odd divisible
$p \in S$ $\Gamma_v / p\Gamma_v \simeq \mathbb{Z}/p\mathbb{Z}$	HRRC field of type $\{2\}$
closed for an higher level ordering	

$\downarrow$ $\searrow$ $\swarrow$ $\downarrow$

Real-closed field	Chain-closed field
$\emptyset$-generalized real-closed	$\{2\}$-generalized real-closed
Γ_v divisible	$\Gamma_v / 2\Gamma_v \simeq \mathbb{Z}/2\mathbb{Z}$
closed for a usual order	closed for an ordering of level 2^k

Examples of HRRC fields

$R((\Gamma)) = \{\sum_\gamma a_\gamma t^\gamma \mid \gamma \in \Gamma, a_\gamma \in R\}$

support of $\sum_\gamma a_\gamma t^\gamma$ well ordered, R a real-closed field, Γ a totally ordered abelian group

In $K = R((\Gamma))$ define :

- product $t^\gamma t^\delta = t^{\gamma+\delta}$

- sum $\sum_\gamma a_\gamma t^\gamma + \sum_\delta b_\delta t^\delta = \sum_\alpha (a_\alpha + b_\alpha) t^\alpha$

- order $\sum_\gamma a_\gamma t^\gamma >_K 0 \Leftrightarrow a_m >_R 0$

where $m = \min(\text{support} \sum_\gamma a_\gamma t^\gamma)$

- valuation $v : R((\Gamma)) \to \Gamma$

defined by $v(\sum_\gamma a_\gamma t^\gamma) = m = \min(\text{support} \sum_\gamma a_\gamma t^\gamma)$

Then $R((\Gamma))$ is an HRRC field, admitting v as henselian valuation with real-closed residue field R and value group Γ .