The nested loop approach to the O(n) model on random maps

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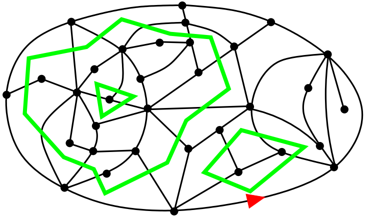
Plan

Introduction

Many statistical physics models can be reformulated in terms of "loop gases" : polymers, self-avoiding walks, percolation, Ising/Potts... and of course $O(n)$ models where n plays the role of a loop fugacity.

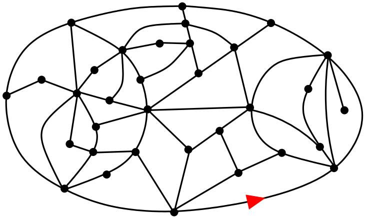
This model is naturally defined on random maps (aka dynamical random lattices). On triangulations, the model was solved via matrix integral techniques [Kostov, Staudacher, Eynard, Zinn-Justin, Kristjansen...].



This solution consists in the computation of the partition function and other "global" quantities, but little is known on the "local" geometry...

Introduction

In contrast, the geometry of random maps without loops is now better understood.



For many map ensembles, the typical graph distance between vertices scales as $m^{1/4}$, where m is the map size. The scaling limit is the Brownian map [Le Gall, Miermont...].

It is unclear how to extend this construction to models with matter. Le Gall and Miermont also studied models of maps with "large" faces, and found possible different scaling limits. We will see that these are related with the O(n) loop model.

Plan

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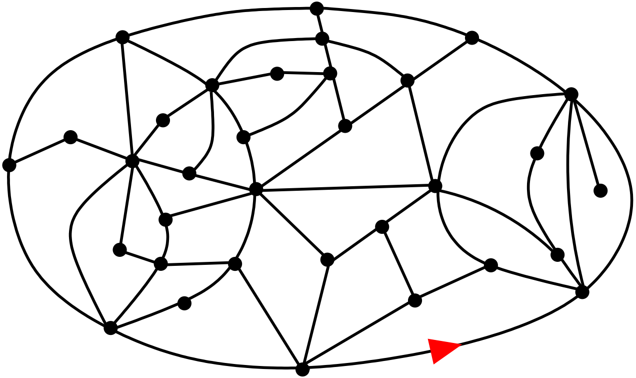
A rooted planar map is a graph embedded in the plane, considered up to continuous deformation, with a distinguished root edge incident to the outer face.



A quadrangulation with a boundary (each inner face has degree 4)

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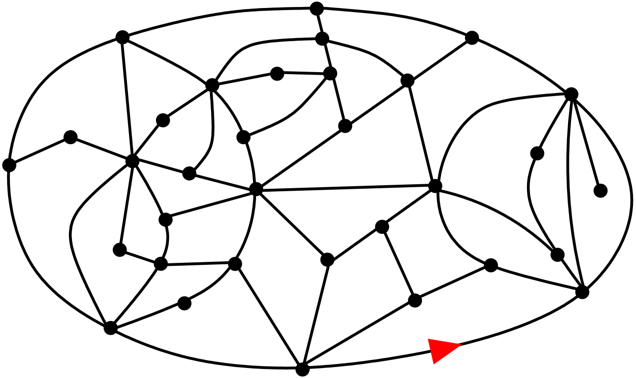

A quadrangulation with a boundary (each inner face has degree 4)

Natural probability measures over maps : uniform distribution over maps with m edges, over triangulations with m triangles, over quadrangulations with m squares...

Boltzmann ensemble of maps with controlled face degrees

(related to the Hermitian one-matrix model)

Partition function

$$\mathcal{F}_p(g_1, g_2, \ldots) = \sum_{\substack{\text{maps with} \\ \text{outer degree } p}} \prod_{k \geq 1} g_k^{\#\{\text{inner faces of degree } k\}}$$

By convention $\mathcal{F}_0(g_1, g_2, \ldots) = 1$ (vertex-map).

Specializations

- Triangulations : $g_k = g$ if $k = 3$, 0 otherwise.
- Quadrangulations : $g_k = g$ if $k = 4$, 0 otherwise.
- Maps with a controlled number of edges : $g_k = t^{k/2}$

Dichotomy

A non-negative weight sequence $(g_1, g_2, \ldots)$ is either :

- non-admissible : $\mathcal{F}_p(g_1, g_2, \ldots) = \infty$ for some p

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But do non-generic critical weight sequences exist ?

Le Gall-Miermont construction

Pick a reference sequence $(g_1^\circ, g_2^\circ, \ldots)$ such that

$$g_k^\circ \sim_{k\to\infty} k^{-a}, \qquad a \in (3/2, 5/2).$$

There exists unique constants A, B such that the weight sequence $g_k := A\,B^k\,g_k^\circ$ is non-generic critical and then

$$\mathbb{P}(\text{degree of a typical face} > k) \sim \text{cst.}k^{-a+1/2}$$

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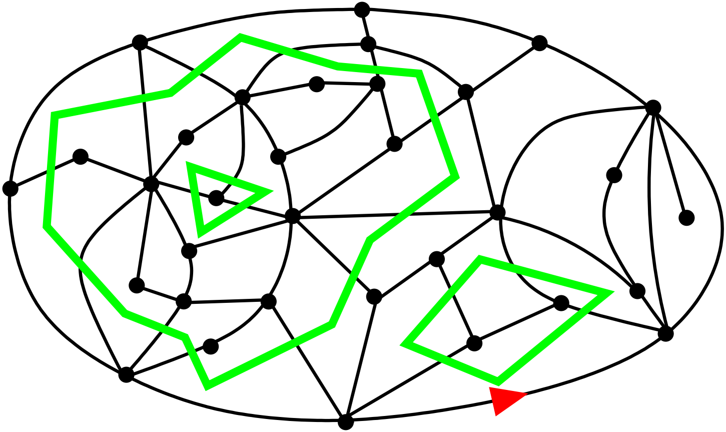
Conditioning the map to have a large number m of vertices, the typical distance between vertices is of order $m^{1/(2a-1)}$ (instead of $m^{1/4}$ for generic critical sequences). This yields a non-generic scaling limit : a "stable" map of Hausdorff dimension $2a-1$, instead of the Brownian map (dimension 4). Is there a "physical" mechanism to produce such non-generic critical sequences ?

Loops

We consider self and mutually avoiding loops on the dual map (by convention, the outer face is not visited).



Each face is incident to 0 or 2 covered edges.

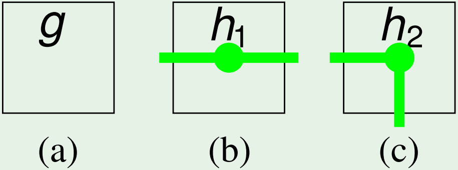
O(n) loop model

Each configuration (map with loops) receives a weight

$$n^{\#\{\text{loops}\}} \times (\text{local weights})$$

Examples

- O(n) loop model on triangulations : weight g per empty triangle, h per visited triangle.
- O(n) loop model on quadrangulations :

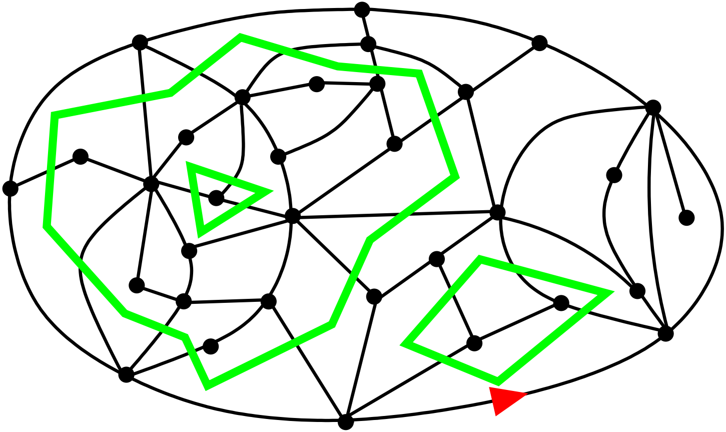


Special cases : rigid case $h_2 = 0$, twisting case $h_1 = 0$.

The partition function of all such models is a specialization of $\mathcal{F}_p(g_1, g_2, \ldots)$!

Plan

The gasket decomposition



Start with a configuration of the O(n) loop model.

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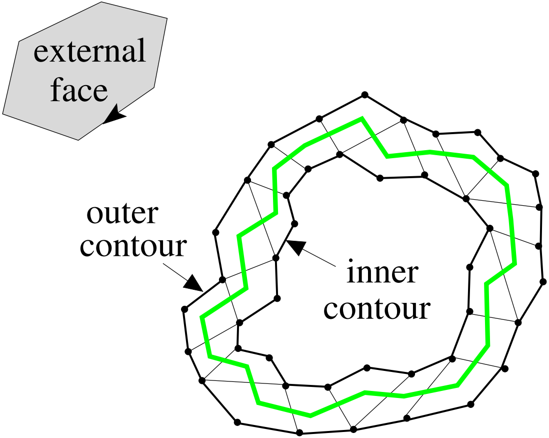
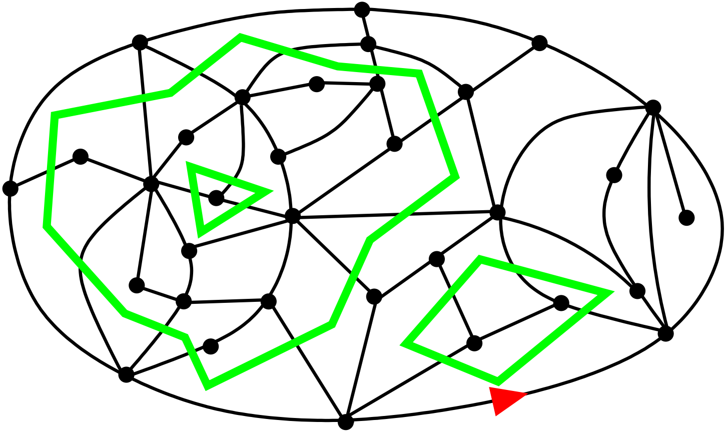


The faces visited by a loop forms a necklace.

The gasket decomposition



Cut along the outer and inner contours of each outermost loop.

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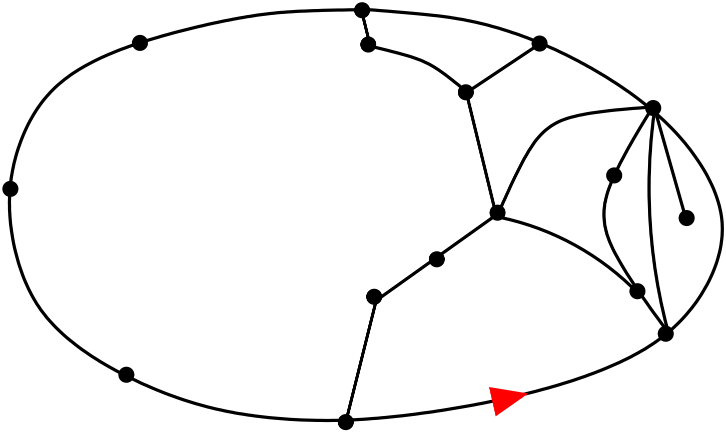

The outer component is the gasket. It is a map without loops, with the same outer degree as the original map.

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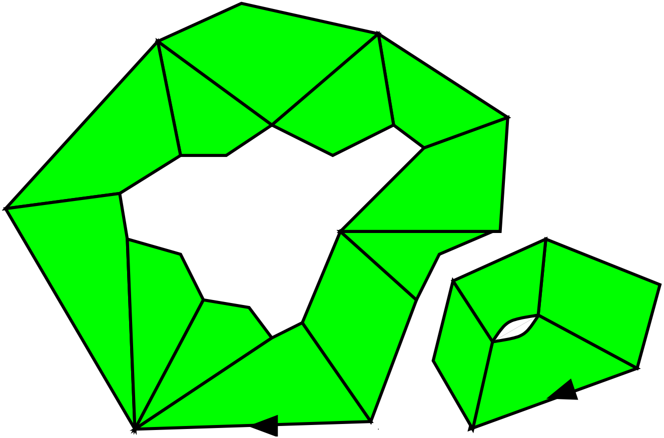
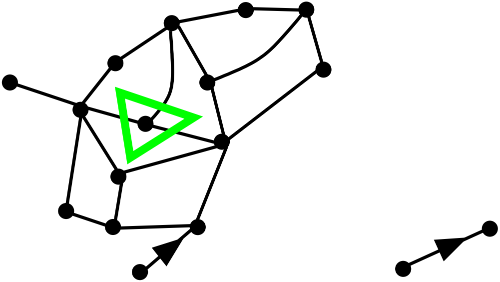


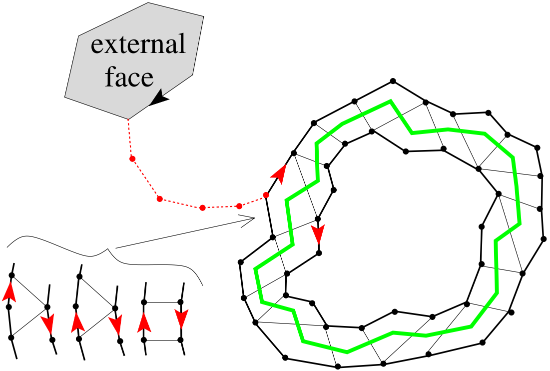
Each outermost loop forms a necklace (cyclic sequence of polygons glued side-by-side).

The gasket decomposition



Each outermost loop contains an internal configuration (of the same nature as our original object).

The gasket decomposition



There exists a well-defined rooting procedure :

- necklaces have a distinguished edge on the outer contour,
- internal configurations are rooted.

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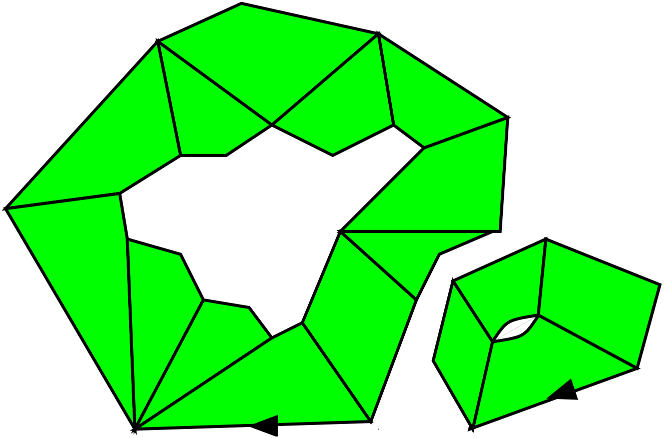

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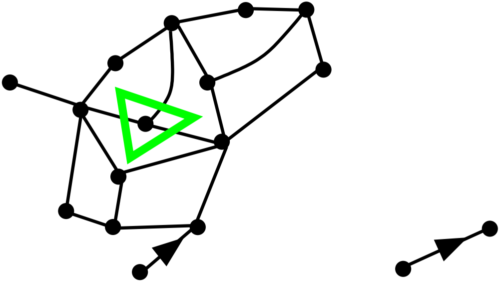

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The gasket decomposition

Bijection

$$\{\text{configurations}\} \simeq \{(\text{gasket, necklaces, internal configurations})\}$$

- A gasket is a map whose faces are either regular faces or holes.
- Each hole of degree $k \geq 1$ is associated with a necklace of outer length k.
- Each necklace of inner length $k' \geq 0$ is associated with an internal configuration of outer degree k'.

The gasket decomposition : consequences

Assumption

Suppose that the weight of a configuration is of the form

$$n^{\#\{\text{loops}\}} \prod_{k\geq 1} \left(g_k^{(0)}\right)^{\#\{\text{empty faces of degree } k\}} \prod_{\text{necklaces}} f(\text{necklace})$$

We denote by

$$F_p = F_p(n; g_1^{(0)}, g_2^{(0)}, \ldots; f)$$

the sum of weights of all configurations with outer degree p. By convention $F_0 = 1$.

Introduce the necklace generating function

$$A(x,y) = \sum_{k\geq 1} \sum_{k'\geq 1} A_{k,k'} x^k y^{k'} := \sum_{\text{necklaces}} f(\text{necklace}) x^{\text{outer length}} y^{\text{inner length}}.$$

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$F_{k'}$

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$$n \sum_{k' \geq 0} A_{k,k'} F_{k'}$$

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$$\mathcal{F}_p(g_1, g_2, \ldots)$$

$$g_k = g_k^{(0)} + n \sum_{k' \geq 0} A_{k,k'} F_{k'}$$

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The gasket decomposition : consequences

Proposition [BBG 2012]

The partition function of our O(n) loop model is obtained from the generating function for maps with controlled face degrees via

$$F_p = \mathcal{F}_p(g_1, g_2, \ldots)$$

where the g_k's satisfy the fixed-point condition

$$g_k = g_k^{(0)} + n \sum_{k' \geq 0} A_{k,k'} \mathcal{F}_{k'}(g_1, g_2, \ldots).$$

Probabilistic interpretation

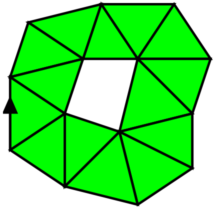
The gasket is distributed according to the Boltzmann measure with face weights $g_1, g_2, \ldots$.

We'll see that critical loop models yield a non-generic weight sequence.

Examples

- O(n) loop model on triangulations

$$A_{k,k'} = \binom{k+k'-1}{k} h^{k+k'}$$



$k = 10, \ k' = 4$

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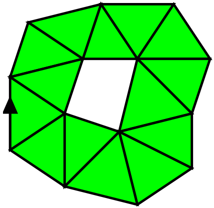

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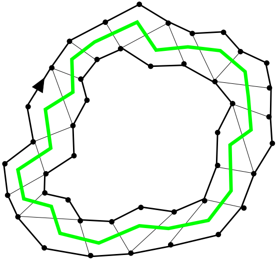
Examples

- O(n) loop model on quadrangulations

$$A_{k,k'} = \sum_{j \equiv k \bmod 2} \frac{2k}{k+k'} \binom{\frac{k+k'}{2}}{j, \frac{k-j}{2}, \frac{k'-j}{2}} h_1^j h_2^{\frac{k+k'}{2}-j}$$

(vanishes for $k + k'$ odd)



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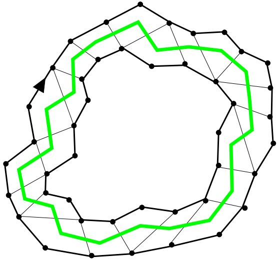

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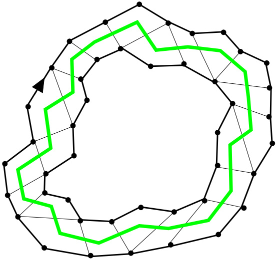

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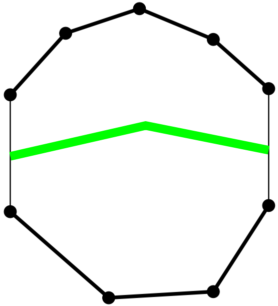
Special cases :

- rigid case $h_2 = 0$: $A_{k,k'} = h_1^k \delta_{k,k'}$
- twisting case $h_1 = 0$: $A_{2k,2k'} = 2\binom{k+k'-1}{k} h_2^{k+k'}$

Examples

- O(n) loop model with general face weights

Attach a weight $h_{\ell,\ell'}$ to each visited face with ℓ (resp. ℓ') edges incident to the outer (resp. inner) contour. In/out symmetry : $h_{\ell,\ell'} = h_{\ell',\ell}$.



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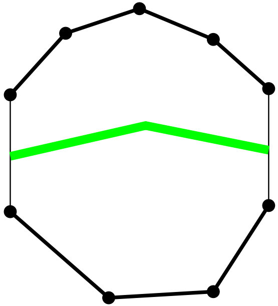
By elementary generatingfunctionology

$$A(x,y) = x\frac{\partial}{\partial x}\log H(x,y)$$

where

$$H(x,y) = \frac{1}{1-\sum_{\ell,\ell'} h_{\ell,\ell'} x^{\ell} y^{\ell'}}.$$



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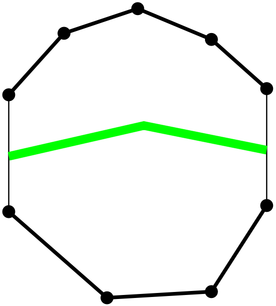

face with weight $h_{4,3}$

Triangular case : $h_{1,0} = h_{0,1} = h$, all other zero.
Quadrangular case : $h_{1,1} = h_2$, $h_{2,0} = h_{0,2} = h_2$, all other zero.

Plan

The resolvent

$$\mathcal{W}(x) := \sum_{p \geq 0} \frac{\mathcal{F}_p(g_1, g_2, \ldots)}{x^{p+1}} \quad \text{(maps with controlled face degrees)}$$

$$W(x) := \sum_{p \geq 0} \frac{F_p(n; \ldots)}{x^{p+1}} \quad \text{(O}(n)\text{ loop model)}$$

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One-cut lemma

For any admissible sequence $(g_1, g_2, \ldots)$, $\mathcal{W}$ defines an analytic function on $\mathbb{C} \setminus [\gamma_-, \gamma_+]$ where $|\gamma_-| \leq \gamma_+$. The "spectral density"

$$\rho(x) := \frac{\mathcal{W}(x - i0) - \mathcal{W}(x + i0)}{2i\pi}$$

is positive and continuous on $]\gamma_-, \gamma_+[$ and vanishes for $x \to \gamma_\pm$.

The resolvent

Functional equation for maps with controlled face degrees

The resolvent is determined by

$$\mathcal{W}(x+i0)+\mathcal{W}(x-i0)=x-\sum_{k\geq 1} g_k x^{k-1}, \qquad x\in[\gamma_-,\gamma_+]$$

and the condition $\mathcal{W}(x)\sim 1/x$ for $x\to\infty$.

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and the condition $\mathcal{W}(x) \sim 1/x$ for $x \to \infty$.

The resolvent of the O(n) loop model is obtained by making the g_k's satisfy the fixed-point condition :

$$W(x+i0)+W(x-i0) = x - \sum_{k\geq 1} g_k^{(0)} x^{k-1} - n \sum_{k\geq 1}\sum_{k'\geq 0} A_{k,k'} x^{k-1} F_{k'}$$

$$= V_0'(x) - \frac{n}{2i\pi} \oint A(x,y) W(y) dy$$

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[BBG 2012]

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[BBG 2012]

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[BBG 2012]

- O(n) model with general face weights : many poles...

The one-pole case

Suppose that $A(x, y)$ is rational with a single pole in y at $y = s(x)$
(as in the triangular and rigid quadrangular cases)

- In/out symmetry implies that s is a homographic involution :

$$s(x) = \frac{\alpha - \beta x}{\beta - \delta x}.$$

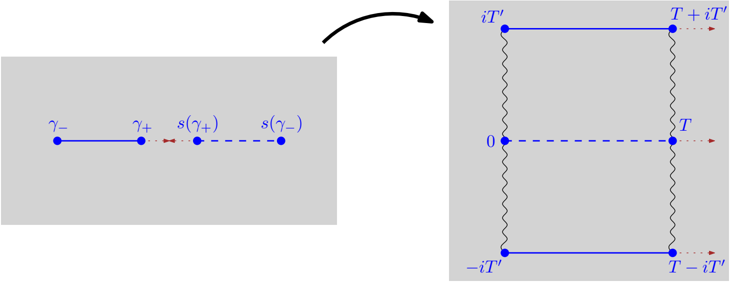
- This situation is generically realized in a model with loop bending energy.
- The functional equation reads

$$W(x + i0) + W(x - i0) - ns'(x)W(s(x)) = V_0'(x) - \frac{ns''(x)}{2s'(x)}$$

whose solution can be explicited using elliptic functions à la Eynard-Kristjansen.

The one-pole case : solution

Introduce a conformal mapping to the torus.



The homogeneous functional equation becomes

$$\omega(v+iT')+\omega(v-iT')=n\,\omega(v)$$

with ω odd and $2T$-periodic.

Non-generic critical points : γ_+ fixed point of s, $T\to\infty$, $T'=\pi$

$$\omega(v)\propto e^{-(2\mp b)v}, \qquad \pi b=\arccos\left(\frac{n}{2}\right), \qquad n\in(0,2)$$

The one-pole case : solution

Returning to the x-plane, this implies that W has a dominant singularity of the form

$$W(x) \propto (x - \gamma_+)^{1 \mp b}, \qquad x \to (\gamma_+)^+$$

hence by transfer

$$\mathbb{P}(\text{degree of a typical gasket face} > k) \sim \text{cst.} k^{3/2 \mp b}.$$

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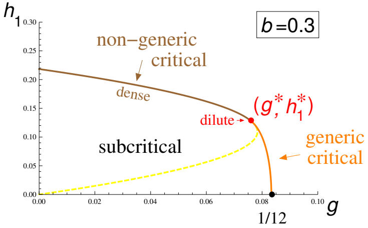
The non-generic critical points forms a "line" in the "phase diagram".

- Unless an extra cancellation occurs, the dense exponent $3/2 - b$ dominates.
- Only at one point, we obtain the dilute exponent $3/2 + b$.
- There is also a generic critical line (as in maps without loops).



Conclusion

Summary

We have shown that the gasket of a critical O(n) loop model has a non-generic critical Boltzmann map distribution. The corresponding stable map has Hausdorff dimension

$$d_H = 3 \pm \frac{2}{\pi} \arccos\left(\frac{n}{2}\right), \qquad n \in (0, 2).$$

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Open questions and directions

- Understand the full scaling limit (not just the gasket), hulls...
- Fully explore the phase diagram of the model with bending energy
- Extend the nested loop approach to other models : Potts, 6-vertex, ADE...