Ref: "The geometry of optimal transportation."
Five lectures on optimal transportation: geometry, regularity and applications.
A glimpse into the differential topology and geometry of optimal transport.

(These three references could be found on McCann's Website.)
www.math.toronto.edu/mccann/publications

\[ c(x, y) = \text{cost per unit being transported from } x \text{ to } y \]
\[ M^+ \subseteq \mathbb{R}^{m+}, \text{ manifolds} \]
\[ d\mu^\pm = f^\pm dx \]

Merge: \( F : M^+ \to M^- \), s.t. \( F \# \mu^+ = \mu^- \) (i.e. \( \mu^+[F(N)] = \mu^-[V] \forall \text{V} \subseteq \mathbb{R}^m \))
(Note: if \( F \) is a diffeo., then \( \det DF(x) = \frac{f'(x)}{f'(F(x))} \))

The problem is:
\[ \inf_{F \# \mu^+ = \mu^-} \int_{M^+} c(x, F(x)) \, d\mu^+ \]

Kantorovich (1942): seek \( \gamma \in \Gamma = \Gamma(\mu^+, \mu^-) \)
\( \gamma \geq 0 \) on \( N = M^+ \times M^- \)
\[ \mu^+(N) = \gamma((1 \times M^+) \cup V) \]
\[ \mu^-(N) = \gamma(M^+ \times 1 \cup V) \]

(e.g., if \( F \# \mu^+ = \mu^- \), then \( \gamma = (id, F) \# \mu^+ \in \Gamma(\mu^+, \mu^-) \))

S.t. \( \inf_{\gamma \in \Gamma} \text{cost}(\gamma) = \inf_{\gamma \in \Gamma} \int_{M^+ \times M^-} c(x, y) \, d\gamma(x, y) \)
\[ \text{minimization of linear programming} \]
over a convex domain.
\( c \in C(M^+xM) \), \( \Gamma \subseteq C(M^+xM)^* = M(M^+xM) \)

(\( F \) is weak-* compact in \( M(M^+xM) \) by Banach-Alaoglu Thm)

- Minimizer exists and Issues:
  1. Unique?
  2. Solve Monge's problem?
  3. Characterization?
  4. Further geometric + analytic properties?

Model case: Brenier (1987), etc. For \( c(x,y) = |x-y|^2 \) and \( \mu^+ = \mu^- \)

\( \text{Thm. } V_{\mu^-} \exists \text{ convex function } u : \mathbb{R}^n \rightarrow [0, \infty] \),

s.t. \( F(x) = Du(x) \), satisfies \( F#\mu^+ = \mu^- \) that

is unique \( \mu^- \)-a.e. and uniquely solves

Monge's and Kantorovich problem.

Regularity of \( F \) for \( c(x,y) = |x-y|^2/2 \).

Delanoë (n=2), Caffarelli (91-96), Urbas (1997)

\( \text{Thm. } \text{If } du^+ = f^+ \text{d}x, M^+ \subseteq \mathbb{R}^n, M^- \text{ convex, } logf \in L^\infty(M^+) \cap C^0 \)

then \( F = Du \in C^d(\mathbb{R}^{\infty}) \) for some \( d \gg \)

for general cost, Ma-Trudinger-Wang

\( L^\infty \Rightarrow C^d \), Figalli-Kim-McCammon.
\( y \) is extremal in \( P \) unless it's the midpoint of a segment in \( P \).

\[ \text{e.g. } \text{id} \# m^* \text{ is extremal in } P, \text{ but not all extremal points have this form.} \]

\[ \text{spt } y = S \subseteq N := M^+ \times M^- \]

\[ \text{smaller, closed set in } N, \text{ carrying full mass for } y. \]

If \( c \in C^2 \) near \((x_0, y_0)\), then \( y \)'s local topology determines dimension of spt \( y \) nearby.

\[ S(x, y; x_0, y_0) := -c(x, y) - c(x_0, y_0) + c(x, x_0) + c(x_0, y) \text{ on } N = M^+ \times M^- \]

Observe: if \( y \) is optimal, then \( S \geq 0 \) on \( S^2 \), where \( S = \text{spt } y \).

Fix \((x_0, y_0) \in S = \text{spt } y \).

Set \( S_o(x, y) = S(x, y; x_0, y_0) \).

Note \( D(x, y) S_o \big|_{(x_0, y_0)} = 0 \).

Define \( h_{y_0} = \text{Hess}_{(x, y)} S_o \big|_{(x_0, y_0)} \).

Taylor expansion:

\[ S_o(x + \alpha x, y + \alpha y) = S_o(x, y) + \frac{1}{2} \alpha (\alpha x, \alpha y) h_{y_0}(\alpha y) + \text{h.o.t} \]

\textbf{Symmetry}

\( \text{sgn}(h) = (h_+, h_0, h_-), \text{ h}_+ + h_+ + h_- = n_+ + n_- \)

\[ h = \begin{pmatrix} 0 & D_{xyc} \\ D_{xyc}^T & 0 \end{pmatrix} \]

\[ \text{if } h_o(\alpha x) = \lambda(\alpha y), \lambda > 0, \Rightarrow \begin{cases} h_+ + h_- = n_+ + n_- - \lambda \alpha y \\ h_+ = h_-, \frac{1}{2} \lambda \alpha y \end{cases} \]

\[ \Rightarrow \begin{cases} h_0(\alpha x) = \lambda(\alpha y) \\ \text{e.g. } n_+ = n_- \Rightarrow h_0 = 0 \end{cases} \]
Ihm (Pass, McCann - Pass - Wan) 

Suppose $c \in C^2(M^+ \times M^-)$, $u^2$ compactly supported, $y$ be a solution of Kantorovich problem. Suppose $(u(x), y(x)) \in S^{n+1}$ and $c$ is non-degenerate at $(x_0, y_0)$. Then there is a neighborhood $N$ of $(x_0, y_0)$, such that $N \cap S^{n+1}$ is contained in an $n$-dimensional Lipschitz submanifold. In particular, if $D^2_x c$ is nonsingular everywhere, $spt \, y$ is contained in an $n$-dimensional Lipschitz submanifold.

Sketch of proof

Does Monge's problem have a solution? $\mu \ll H^n$ by Gangbo (1996), Lévin (1999). Yes if $c \in C^1(M^+ \times M^-)$, $x \mapsto c(x, y)$ is a map from $M^+ \to S_0^{n-1}(xy)$. Then $spt \, p \subseteq \text{Graph}(F)$ for some $F: M^+ \to M^-$. 

E.g. $c(x, y) = |x-y|^p$, $p > 1$, on $M^+ = \mathbb{R}^n$, for $p = 2$, $S_0^{n-1}(xy)$. Notice $(A_1)$ can't be satisfied if $M^+$ is compact, possible solutions: relax $C^1$ hypothesis. E.g. $c = d^2$ on any Riemann manifold. $(M^+ = (M, g))$. E. Monge solution $\mu$ unique (McCann 2001)
Thin (Chiappori - McCann - Nesheim)

c \in C^1, \mu^+ \ll dx, \text{ if } \forall x \in M^+, y \neq x \in M^-.

x \in M^+ \rightarrow So(x, y) has at most 2 critical points (a global min & max) then the Kantorovich solution
is unique (but may not be Menge).

open question: can such a condition exist \( M^+ = \mathbb{T}^2 \) or
other topology \( \pi_i \)?

Regularity for general cost requires

\((A_0)\) \quad c \in C^4 (N) \quad \forall (x_0, y_0) \in N.

\((A_1)\) \quad y \in M^- \rightarrow D_x c(x_0, y) \text{ and } x \in M^+ \rightarrow D_y c(x, x_0) \text{ are injective.}

\((A_2)\) \quad \det D^2 c(x_0, y_0) = \det (c_{ij}) \neq 0.

\((A_3)\) \quad \text{cross} (p, q) \geq 0 \quad \text{for all } (p, q) \in T(x_0, y_0) M^+ \times M^- \text{ such that } p^i c_{ij} q^j \geq 0.

\((A_4)\) \quad M^-_{1,0} := D_x c(x_0, M^-) \subseteq \mathbb{R}^n \quad \text{and} \quad M^+_1 := D_y c(M^+, x_0) \subseteq \mathbb{R}^n \text{ are convex.}

where \text{cross} (p, q) := \text{see}^N_{x_0, y_0} p \oplus 0 \wedge 0 \oplus q

depends on the secondary curvature of the semi Riemannian metric \( h \) on \( N = M^+ \times M^- \).