MINIMAL SUBMANIFOLDS AND LOWER CURVATURE

Yesterday: focused on codim 1 case. Now we extend.

2 dimnl case: interesting methds via complex geometry

\[ \Sigma \to M, v \in \Gamma(NM) \text{ normal bundle, assume } M \text{ orientable, } x^1, x^2, x^3 = x^0, i x^0 \]

\[ \Sigma^2 \Sigma(v, v) = \int_{\Sigma} \left[ \left| \nabla_{\bar{z}}^2 v \right|^2 - \left| \nabla_{\bar{z}}^4 v \right|^2 - R(u_2, v, u_3, \bar{v}) \right] d\sigma d\tau \]

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We rewrite \( \Sigma^2 \Sigma \) in a better way to understand its geometric meaning via complex coordns.

\[ \text{each term has conformormal invariance.} \]

\[ \text{IBP to get rid of } z \text{ derivatives} \]

\( (z \text{ either complex, } v, u \text{ on } \Sigma \text{ do not bdy term}) \)

\[ \text{(idea from page 1988 Micaleff & Moore)} \]

\[ \text{(\textbf{Pic}) if } K_{\pi} > 0 \text{ for all isotropic } \pi. \]

\[ \text{Theo (Micaleff & Moore) Geometric Thm: If } M \to M \text{ minimal (harmonic) } \]

\[ \text{then } Ind(u(S^2)) > \left[ \frac{1}{2} \right] \]

\[ \text{pf uses special facts about holomorphic bundle over 2-sphere.} \]

\[ \text{In 2-D have integrability on normal bundle} \]

\[ \text{idea: Construct sufficiently many } \psi \in \Gamma(N_0 \Sigma) \text{ with } \nabla_{\bar{z}}^2 \psi = 0 \text{ \& } \langle \psi, \psi \rangle = 0. \]

\[ \text{Combine with topolgical results to obtain} \]

\[ \text{Cor: Topolgical sphere thm for punctual quarter pinching} \]

\[ \text{Note "pinching" means } \forall \psi \text{ on } M \text{ with } K_\pi > 0 \text{, have } \max K_\pi < \psi. \]

\[ \text{(Brendle-S) } \text{Pic is preserved under Ricci flow.} \]

\[ \text{Pic} \leq M \times \mathbb{R} \text{ Pic} \leq M \times \\text{Pic} \geq \text{ conjecture} \text{ pinching} \]

\[ \text{2 conditions related to Pic:} \]

\[ \text{M is a spherical space form} \]
Q: what about PIC?

Thm (Micallef-Wang) If $M_1, M_2$ are PIC then $M_1 \# M_2$ has PIC metric

$$S^n \setminus \Gamma, S^{n-1} \times S^1.$$ (remember, $n > 4$)

get "handle-bodies". Freegps

PIC conj: If $M$ is PIC then $M$ is virtually covered by $(S^{n-1} \times S^1) \# \cdots \# (S^{n-1} \times S^1)$

In dim 4 the Ricci flow for PIC is special.


PIC conjecture holds.

Next: $T^1$ PIC conjecture. If $M$ is PIC $\Rightarrow$ $\pi_1(M)$ is virtually free.

(there is free subgroup in $\pi_1$ which has finite index).

Thm (Gadgil, Seshadri) If $M$ is PIC and has free $\pi_1$ then $M$ is homeomorphic to

$$(S^{n-1} \times S^1) \# \cdots \# (S^{n-1} \times S^1)$$

There is a minimal surface approach to this.

Geometric Question: $\tilde{M}$ universal cover.

$M$ has finite fill radius $\tilde{M}, \exists \Sigma$ disk $\exists \Sigma \subset \tilde{M}$, filled disk $\exists \Sigma \subset \tilde{M}$ distance nbhd of $\Gamma$

Thm (M. Ramachandran, J. Wolf 2010). If $M$ has finite fill radius then $\pi_1(M)$ is virtually free.

$\Gamma \in \tilde{M}_1, \exists \Sigma$ least area dihedral $\exists \Sigma \subset \Gamma$.

Hope: $\Sigma \leq N_c(\Gamma)$ for some $c$ fixed.

True when $\dim M = n = 3, R_M > X$. A Bonnet type thm occurs

$\Sigma^* \subset M$ stable $\Rightarrow$ diam$(\Sigma^*) \leq \pi \sqrt{X}$.

Minimal surface conjecture:

$$M, IC > X \implies \exists \Sigma \subset \tilde{M}$$ stable minimal disk,

then $\forall$ $p \in \Sigma$, $d(p, \partial \Sigma) \leq c/\sqrt{X}$.

Remarks on Conj.

2nd Variation: $X \int_{\Sigma} \nabla^2 V \, d\mu - \int_{\Sigma} \nabla^2 V \, d\mu$ one way to get rid of $\text{term}$ to construct hole into sections.

If disk is quite large, hence eigenvalue to be small.

Idea: $V$ hole, isotropic, slow growth.

Thm (A. Fraser, 2003) If $M$ is PIC then $\pi_1(M)$ contains no copy of $\mathbb{Z} \times \mathbb{Z}$.

So torsion subgps can be represented by minimal tori.

Stability fn $T^2 \to S^{n-1} \times S^1 \to \mathbb{S}^1 \times S^1$. 
The Theorem \( \Rightarrow \)'s

If \( \mathcal{U} : \mathbb{T}^3 \to M \) PIC, minimal then \( \mathcal{U} : (k \mathbb{T}^3) \to M \) unstable for \( k \) large.

Notion: stable homology, from alg. geom. If you have a homology class representable by a calibrated/complex submanifold then every covering is minimizing.

Fraeher: although you may have stable torus, for sufficiently high covering, get instability.

Basic Idea: Construct approx holom. isotropic \( V \) on \( k \mathbb{T}^3 \).

\[
\int_{k\Sigma} |\nabla_{\Sigma} V|^2 \leq \varepsilon(k) \int_{k\Sigma} |V|^2.
\]

Analog. to Gromov & Lawson idea.

Takes, for \( k \) large, the large torus will have a contracting map to \( S^3 \); then can take a line bundle on \( S^3 \) and pull it back, tensor it to a complexified normal bundle, this has a holomorphic section. Use half bundle with fact that map is holomorphic, you can make it almost holomorphic.
Geometric Result for $\mathbb{E}^k \rightarrow \mathbb{M}^3$, $R > 1$:

- $\mathbb{M}$ has bounded fill radius
- There exists $\mathcal{D} \subseteq \mathbb{M}^3$ such that $D \subseteq N_{\frac{R}{2}}(\mathcal{D})$

**Idea:** $u_0 > 0$, $L u_0 \leq 0$.

- Weighted arclength $L_{u_0}(\gamma) = \int_0^1 u_0 \cdot ds$.
- Surface Bonnet-type property if $R > 1$.
- With direct measured in $g$.

$\Sigma = \mathbb{M}^3$:

- Complex form: $(\cdot, \cdot)$ complex linear
- $N_\Sigma$ $\langle \cdot, \cdot \rangle$ hamiltoni pacein
- $\Sigma \times \Sigma \rightarrow \mathbb{R}^\omega$ complex
- Linear

**Index form:** $w \in \mathfrak{P}(N_\Sigma)$.

$I(w, w) = \int_\Sigma \left[ |D_w^2 w|^2 - R(w, u_{\Sigma}, \overline{w}, u_{\Sigma}) - |(D^2 w)|^2 \right]$
Micalef-Moore: \((U_2, U_2) = 0\)

Isotropic

\(\text{Def: } M \text{ has positive isotropic curvature} \)

\(K(\pi) > 0 \ \forall \ \pi \ \text{isotropic 2-disc complex}\)

\(S^2 \to M^n \ (\text{Pic})\)

\(\text{ind} \geq \left\lfloor \frac{n-2}{2} \right\rfloor\)

\(T_1(M) = S^{13} \ (\text{Pic}) \Rightarrow M \text{ homeo to } S^n\)

Case: \(K > 0 \ \forall \ P\)

\(\frac{\max K(\pi)}{\min K(\pi)} < 4\)

\(T_1 = S^{13}\)

\(\Rightarrow M \cong S^n\)

S. Brendle - S: Can improved to differ without simply connected:

\([\text{Ricci Flow}] = M \times \mathbb{R} \oplus M \times \mathbb{R}^2\)

Topological Classification of P/I C:

\(n = 4, \ R. \ Hamilton, \ B. \ Chen - X. \ Zhu\)

\(n > 5 ?\)
Route to topological classification:

Conjecture: \( \Pi_1 \) is virtually free.

Thm: (M. Ramachandran, J. Watson)
Conjecture follows from \( \tilde{M} \) has bounded full radius.

Thm: (S. Gadgil, H. Seshadri 2009) If \( M \) is in \( \text{PI}_2 \) and \( \Pi_1(M) \) is a free group then \( M \) is homeomorphic to connected sum of \( h \) copies of \( S^1 \times S^{n-1} \).

Conjecture: \( (\Sigma, \partial \Sigma) \preceq M \) \( K(\Pi) \geq 1 \).

\[ \Sigma \preceq N_c(\partial \Sigma) \]