# Metric measure spaces with Ricci lower bounds, Lecture 2

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# Isoperimetric problem

One of oldest problems in mathematics, roots in myths of 2000 years ago (Queen Dido's problem). Roughly 3 questions:

Q1 Given a space $X$ what is the minimal amount of area needed to enclose a fixed volume $v > 0$?

Q2 Is there an optimal shape?

Q3 Describe/characterize the optimal shapes.

# Examples

Not many examples of spaces $X$ where we can fully answer Q1,Q2,Q3:

- $X = \mathbb{R}^n \rightsquigarrow$ only optimal shapes are round balls: $|\partial E| \geq |\partial B|$ where $B$ is a round ball s.t. $|B| = |E|$.
- $X = S^n$ or $X = H^n$ analogous: only optimal shapes are metric balls: $|\partial E| \geq |\partial B|$ where $B$ is a metric ball s.t. $|B| = |E|$
- Not many other examples (e.g. $\mathbb{R}P^3$ by Ritoré-Ros): in general the spaces for which we can fully answer Q1,Q2,Q3 are either very symmetric or perturbations of very symmetric spaces.
- Results in presence of mild singularities but still very symmetric (conical manifolds: Morgan-Ritoré '02, Milman-Rotem '14. Polytopes: Morgan '07).

# Levy-Gromov inequality

Besides the euclidean one, probably the most famous isoperimetric inequality is the Levy-Gromov isoperimetric inequality:

Levy-Gromov Isoperimetric inequality

Let $(M^n, g)$ Riemannian manifold with $Ric_g \geq Kg$, $K > 0$, and $E \subset M$ domain with smooth boundary $\partial E$. Then

$$\frac{|\partial E|}{|M|} \geq \frac{|\partial B|}{|S|} \quad (LGI)$$

where $S = S^n_K$ round sphere with $Ric \equiv K$, and $B \subset S$ is a spherical cap s.t. $\frac{|E|}{|M|} = \frac{|B|}{|S|}$.

# Isoperimetry in m.m.s.

DEF: Let $(X, \mathsf{d}, \mathfrak{m})$ be a m.m.s. with $\mathfrak{m}(X) = 1$ and let $E \subset X$ be a Borel set. Define the outer Minkowski content

$$\mathfrak{m}^+(E) := \liminf_{\varepsilon \to 0^+} \frac{\mathfrak{m}(E^\varepsilon) - \mathfrak{m}(E)}{\varepsilon}$$

where $E^\varepsilon := \{x \in X \ : \ \mathsf{d}(x, E) < \varepsilon\}$

DEF: The isoperimetric profile function $\mathcal{I}_{(X,\mathsf{d},\mathfrak{m})} : [0,1] \to \mathbb{R}^+$ is the largest function such that $\mathfrak{m}^+(E) \geq \mathcal{I}_{(X,\mathsf{d},\mathfrak{m})}(\mathfrak{m}(E))$ $\forall E \subset X$, i.e.

$$\mathcal{I}_{(X,\mathsf{d},\mathfrak{m})}(v) := \inf\{\mathfrak{m}^+(E) \ \text{s.t.} \ \mathfrak{m}(E) = v\}.$$

RK: - Q1 amounts to compute/estimate $\mathcal{I}_{(X,\mathsf{d},\mathfrak{m})}$.
- (LGI) states $\mathcal{I}_{(M^n, \mathsf{d}_g, vol/(vol(M)))} \geq \mathcal{I}_{(S^n_K, \mathsf{d}_{g_{S^n_K}}, vol_{S^n_K}/(vol_{S^n_K}(S^n_K)))}$

Q: - is there an analog of (LGI) for $Ric \geq K$, $K \leq 0$?
-what about the (LGI) in $RCD(K,N)$ spaces?

# Extension of (LGI) by E. Milman to arbitrary weighted manifolds and to $K \in \mathbb{R}$

DEF: Let $(M^n, g)$ be a Riemannian manifold and let $\mathfrak{m} := \Psi\, vol_g$ for some smooth function $\Psi \geq 0$. We say that $(M^n, g, \mathfrak{m})$ satisfies the $CDD(K, N, D)$ condition if

- $\mathrm{supp}(\mathfrak{m}) \subset \Omega$ for some $\Omega \subset X$ geodesically convex with $\mathrm{diam}(\Omega) \leq D$.
- $Ric_{g,\Psi,N} := Ric_g - (N-n)\frac{\nabla^2 \Psi^{1/N-n}}{\Psi^{N-n}} \geq Kg$

THM[E. Milman '12] For every $K \in \mathbb{R}, N, D > 0$ there exists an explicit function $\mathcal{I}_{K,N,D} : [0,1] \to \mathbb{R}^+$ such that if $(M^n, g, \Psi)$ satisfies $CDD(K, N, D)$ then $\mathcal{I}_{(M^n, \mathsf{d}_g, \Psi vol)} \geq \mathcal{I}_{K,N,D}$.

RK: -for $K > 0$, $N = n \in \mathbb{N}$ and $D \geq \mathrm{diam}(S^n_K)$, one re-obtains (LGI) since in this case $\mathcal{I}_{K,N,D} = \mathcal{I}_{(S^N_K, \mathsf{d}_{g_{S^N_K}}, vol_{S^N_K}/(vol_{S^N_K}(S^N_K)))}$.

- for $K \leq 0$ there is not a model space as in (LGI), nevertheless there is a model isoperimetric profile function defined piecewise in an explicit way.

## Extension to $RCD(K,N)$ spaces

THM[Cavalletti-M. '15] Levy-Gromov-Milman isoperimetric inequality holds in $RCD(K,N)$ spaces, i.e. if $(X, \mathrm{d}, \mathfrak{m})$ is an $RCD(K,N)$ space with $\mathfrak{m}(X) = 1$ and $\mathrm{diam}(X) \leq D$ then $\mathcal{I}_{(X,\mathrm{d},\mathfrak{m})} \geq \mathcal{I}_{K,N,D}$.

COR: If $(X, \mathrm{d}, \mathfrak{m})$, with $\mathfrak{m}(X) = 1$, is an $RCD(K,N)$ space for some $K > 0$ and $2 \leq N \in \mathbb{N}$ then (LGI) holds, i.e. for every Borel subset $E \subset X$

$$\mathfrak{m}^+(E) \geq \frac{|\partial B|}{|S|}$$

where $S = S^N_K$ round sphere with $Ric \equiv K$, and $B \subset S$ is a spherical cap s.t. $\mathfrak{m}(E) = \frac{|B|}{|S|}$.

RK: Seems new even for Ricci limit spaces and Alexandrov spaces (sketch of proof by Petrunin in curv$\geq 1$)

# Proof part 1: 1-D localization

Assume for the moment that given $E \subset X$ we can find a "1-D localization" $\{X_q\}_{q\in Q}$ of X, i.e.

1. $\{X_q\}_{q\in Q}$ is a partition of $X$, i.e. $X = \mathring{\bigcup}_{q\in Q} X_q$,
2. $\mathfrak{m} = \int_Q \mathfrak{m}_q \, \alpha(dq)$, with $\alpha(Q) = 1$ and $\mathfrak{m}_q(X_q) = \mathfrak{m}_q(X) = 1$ for $\alpha$-a.e. $q \in Q$
   $\rightsquigarrow$ disintegration of $\mathfrak{m}$ (kind of non-straight Fubini)
3. $X_q$ is a geodesic in $X$ and $(X_q, |\cdot|, \mathfrak{m}_q)$ is a $CD(K, N)$ space
4. $\mathfrak{m}_q(E \cap X_q) = \mathfrak{m}(E)$, for $\alpha$-a.e. $q \in Q$,

RK the first two assumptions are mild, the characterizing properties are the last two.

## Proof part 2: conclusion

If for every given $E \subset X$ we can find a 1-D localization as above then

$$
\begin{aligned}
\mathfrak{m}^+(E) &= \liminf_{\varepsilon \to 0^+} \frac{\mathfrak{m}(E^\varepsilon) - \mathfrak{m}(E)}{\varepsilon} \\
&= \liminf_{\varepsilon \to 0^+} \int_Q \frac{\mathfrak{m}_q(E^\varepsilon) - \mathfrak{m}_q(E)}{\varepsilon} \, \alpha(dq) \quad \text{by 2.} \\
&\geq \int_Q \liminf_{\varepsilon \to 0^+} \frac{\mathfrak{m}_q(E^\varepsilon \cap X_q) - \mathfrak{m}_q(E \cap X_q)}{\varepsilon} \, \alpha(dq) \quad \text{by 2.} \\
&\geq \int_Q \mathfrak{m}_q^+(E \cap X_q) \, \alpha(dq), \text{ by } E^\varepsilon \cap X_q \supset (E \cap X_q)^\varepsilon \cap X_q \\
&\geq \int_Q \mathcal{I}_{K,N,D}(\mathfrak{m}_q(E)) \, \alpha(dq) \text{ by 3.+Smooth LGMI} \\
&= \int_Q \mathcal{I}_{K,N,D}(\mathfrak{m}(E)) \, \alpha(dq) \text{ by 4.} = \mathcal{I}_{K,N,D}(\mathfrak{m}(E)).
\end{aligned}
$$

# Proof part 3: how to construct a 1-D localization

- Recall that $\mathfrak{m}(X) = 1$, fix $E \subset X$ with $\mathfrak{m}(E) \in (0,1)$,
- Let $\mu_0 := \frac{\chi_E}{\mathfrak{m}(E)}\,\mathfrak{m}$ and $\mu_1 := \frac{1-\chi_E}{1-\mathfrak{m}(E)}\,\mathfrak{m} = \frac{\chi_{X\setminus E}}{\mathfrak{m}(X\setminus E)}\,\mathfrak{m}$
- Consider the $L^1$-optimal transport problem

$$\inf \left\{ \int_{X\times X} \mathsf{d}(x,y)\, d\gamma \,:\, \gamma \in \mathcal{P}(X\times X), (\pi_1)_\sharp\gamma = \mu_0, (\pi_2)_\sharp\gamma = \mu_1 \right\}$$

- By Optimal Transport techniques there exists a minimizer $\gamma \in \mathcal{P}(X \times X)$ and a 1-Lipschitz function $\varphi : X \to \mathbb{R}$ called Kantorovich potential such that, denoted

$$\Gamma := \{(x,y) \in X \times X \,:\, \varphi(x) - \varphi(y) = \mathsf{d}(x,y)\},$$

  $\gamma$ is concentrated on $\Gamma$.
- The relation $\sim$ on $X$ given by $x \sim y$ iff $(x,y) \in \Gamma$ or $(y,x) \in \Gamma$ is an equivalence relation on $X$ (up to an $\mathfrak{m}$-negligible subset) and the equivalence classes are geodesics.
  $\rightsquigarrow$ partition of $X$ into geodesics driven by $E$
- More work to prove properties 3. and 4.

# Why $L^1$-trasport?

- It is more standard to consider the $L^2$-optimal tranport problem: given $\mu_0, \mu_1 \in \mathcal{P}(X)$ let
$$\inf \left\{ \int_{X\times X} \mathsf{d}(x,y)^2 \, d\gamma \, : \, \gamma \in \mathcal{P}(X \times X), (\pi_1)_\sharp \gamma = \mu_0, (\pi_2)_\sharp \gamma = \mu_1 \right\}.$$
Which defines a metric $W_2$ on $\mathcal{P}(X)$.
- Now if $(\mu_t)_{t\in[0,1]}$ is a $W_2$ geodesic from $\mu_0$ to $\mu_1$ we know that $\mu_t$ is concentrated on midpoints of geodesics from *supp*$(\mu_0)$ to *supp*$(\mu_1)$:
$\mu_t(\{\gamma(t) \, : \gamma \, \mathit{geod}, \gamma(0) \in \mathrm{supp}(\mu_0), \gamma(1) \in \mathrm{supp}(\mu_1)\}) = 1,$
- moreover, from $\mathsf{d}^2$-monotonicity, if $\gamma_1$ and $\gamma_2$ are such geodesics with $\gamma_1(0) \neq \gamma_2(0)$ then $\gamma_1(t) \neq \gamma_2(t)$ in a.e. sense.
$\rightsquigarrow$ the transport at time $t$ is given by a map (Brenier map).
- BUT it may happen $\gamma_1(s) = \gamma_2(t)$ for $s \neq t$
$\rightsquigarrow$ $L^2$-transport does not induce an equivalence relation.
- On the other hand $L^1$ transport does induce an equivalence relation into rays where the transport is performed
$\rightsquigarrow$ partition of the space into $1D$ objects.

# Brief history of 1-D localization technique

The localization technique is a way to reduce an a-priori complicated high dimensional problem to a simpler 1-dimensional problem.

- In $\mathbb{R}^n$ or $S^n$, using the high symmetry of the space, 1-D localizations can be usually obtained via iterative bisections
  - Roots in a paper by Payne-Weinberger '60 about sharp estimate of $1^{st}$ eigenvalue of Laplacian
  - Formalized by Gromov-V. Milman '87, Kannan - Lovász - Simonovits '95
- Extended by B. Klartag '14 to Riemannian manifolds via $L^1$-optimal trasport: no symmetry but still heavily using the smoothness of the space (estimates on $2^{nd}$ fundamental form of level sets of the Kantorovich potential $\varphi$)
- Extension to non-smooth spaces by Cavalletti-M. '15.

# Rigidity of (LGI)

- It is well known that in smooth setting (LGI) are rigid: if $(M^n, g)$ has $Ric_g \geq (n-1)g$ and if there exists $v \in (0,1)$ such that $\mathcal{I}_{(M^n, \mathsf{d}_g, vol/(vol(M)))}(v) = \mathcal{I}_{(S^n, \mathsf{d}_{g_{S^n}}, vol_{S^n}/(vol_{S^n}(S^n)))}(v)$ then $M$ is isometric to $S^n$.
- Q: is it true also for non smooth spaces?
- NO: spherical suspensions have the same isoperimetric profile function of the round sphere.
- Q: are spherical suspensions the only cases? YES!

THM (Cavalletti-M. '15) If $(X, \mathsf{d}, \mathfrak{m})$ is an $RCD(N-1, N)$ space and there exists $v \in (0,1)$ such that $\mathcal{I}_{(X,\mathsf{d},\mathfrak{m})}(v) = \mathcal{I}_{N-1,N,\pi}(v)$,

Then $(X, \mathsf{d}, \mathfrak{m})$ is a spherical suspension: $X \simeq [0,\pi] \times_{\sin}^{N-1} Y$ as m.m.s. for some $RCD(N-2, N-1)$ space $(Y, \mathsf{d}_Y, \mathfrak{m}_Y)$

Moreover, in this case, the following hold:

*i*) For every $v \in [0,1]$ it holds $\mathcal{I}_{(X,\mathsf{d},\mathfrak{m})}(v) = \mathcal{I}_{N-1,N,\infty}(v)$. $\rightsquigarrow$ Q1

*ii*) For every $v \in [0,1]$ there exists a Borel subset $A \subset X$ with $\mathfrak{m}(A) = v$ such that $\mathfrak{m}^+(A) = \mathcal{I}_{(X,\mathsf{d},\mathfrak{m})}(v) = \mathcal{I}_{N-1,N,\pi}(v)$. $\rightsquigarrow$ Q2

*iii*) If $\mathfrak{m}(A) \in (0,1)$ then $\mathfrak{m}^+(A) = \mathcal{I}_{(X,\mathsf{d},\mathfrak{m})}(v) = \mathcal{I}_{N-1,N,\pi}(v)$ if and only if $\bar{A} = \{(t,y) \in [0,\pi] \times_{\sin}^{N-1} Y \, : \, t \in [0, r_v]\}$ or $\bar{A} = \{(t,y) \in [0,\pi] \times_{\sin}^{N-1} Y \, : \, t \in [\pi - r_v, \pi]\}$, where $\bar{A}$ is the closure of $A$ and $r_v \in (0,\pi)$ is chosen so that $\int_{[0,r_v]} c_N (\sin(t))^{N-1} dt = v$, $c_N$ being given by $c_N^{-1} := \int_{[0,\pi]} (\sin(t))^{N-1} dt$. $\rightsquigarrow$ Q3

# Proof of rigidity

- The idea is to show that $\mathrm{diam}(X) = \pi$ and then apply Cheng-Ketterer Maximal Diameter Theorem which gives that $X$ is a spherical suspension.
- Assume by contradiction there exists $\bar{v} \in (0,1)$ such that $\mathcal{I}_{(X,\mathsf{d},\mathfrak{m})}(\bar{v}) = \mathcal{I}_{N-1,N,\pi}(\bar{v})$ but $\mathrm{diam}(X) \leq \pi - \varepsilon_0 < \pi$.
- Key observation: there exists $\delta > 0$ such that $\mathcal{I}_{N-1,N,\pi}(\bar{v}) \leq \mathcal{I}_{N-1,N,D}(\bar{v}) - \delta$ for every $D \in (0, \pi - \varepsilon_0]$.
- Let now $E \subset X$ be such that $\mathfrak{m}(E) = \bar{v}$ and $\mathfrak{m}^+(E) \leq \mathcal{I}_{(X,\mathsf{d},\mathfrak{m})}(\bar{v}) + \frac{\delta}{2} = \mathcal{I}_{N-1,N,\pi}(\bar{v}) + \frac{\delta}{2}$.
- Arguing as in the proof of the isoperimetric inequality by 1-D localization associated to $E$ we get

$$
\begin{aligned}
\mathcal{I}_{N-1,N,\pi}(\bar{v}) + \frac{\delta}{2} \quad &\geq \quad \mathfrak{m}^+(E) \geq \int_Q \mathfrak{m}_q^+(E \cap X_q)\, \alpha(dq) \\
&\geq \quad \int_Q \mathcal{I}_{N-1,N,|\mathrm{diam}(X_q)|}(\bar{v})\, \alpha(dq) \text{ by } \mathfrak{m}_q(E) = \bar{v} \\
&\geq \quad \mathcal{I}_{N-1,N,\pi}(\bar{v}) + \delta \;\text{ by } \mathrm{diam}(X_q) \leq \mathrm{diam}(X)
\end{aligned}
$$

# Almost rigidity of (LGI)

Q: what happens if $\mathcal{I}_{(X,\mathrm{d},\mathfrak{m})}$ is close to the model $\mathcal{I}_{N-1,N,\pi}$? Does it imply $(X, \mathrm{d}, \mathfrak{m})$ close to a spherical suspension?

THM(Cavalletti-M.'15) For every $N \in [2, \infty)$, $v \in (0, 1)$, $\varepsilon > 0$ there exists $\bar{\delta} = \bar{\delta}(N, v, \varepsilon) > 0$ such that the following hold. For every $\delta \in [0, \bar{\delta}]$, if $(X, \mathrm{d}, \mathfrak{m})$ is an $RCD(N - 1 - \delta, N + \delta)$ space satisfying

$$\mathcal{I}_{(X,\mathrm{d},\mathfrak{m})}(v) \leq \mathcal{I}_{N-1,N,\pi}(v) + \delta,$$

then there exists an $RCD(N - 2, N - 1)$ space $(Y, \mathrm{d}_Y, \mathfrak{m}_Y)$ with $\mathfrak{m}_Y(Y) = 1$ such that

$$\mathrm{d}_{mGH}(X, [0, \pi] \times_{\sin}^{N-1} Y) \leq \varepsilon.$$

RK The almost rigidity seems new even for smooth manifolds: if $(M^n, g)$ has $Ric_g \geq (n - 1 - \delta)g$ and $\mathcal{I}_{(M,\mathrm{d}_g,vol/vol(M))}(v) \leq \mathcal{I}_{N-1,N,\pi}(v) + \delta$ then $(M^n, g)$ is $mGH$-close to a spherical suspension. $\rightsquigarrow$ an example of application of $RCD$ spaces to smooth manifolds with lower Ricci bounds.

- Step 1 making quantitative the arguments of the rigidity theorem we get that the diameter of $X$ must be almost maximal, more precisely: for every $N \in [2, \infty)$, $v \in (0, 1)$, $\eta > 0$ there exists $\bar{\delta} = \bar{\delta}(N, v, \eta) > 0$ such that if $(X, \mathrm{d}, \mathfrak{m})$ is an $RCD(N-1-\delta, N+\delta)$ space satisfying $\mathcal{I}_{(X,\mathrm{d},\mathfrak{m})}(v) \leq \mathcal{I}_{N-1,N,\infty}(v) + \delta$, for some $\delta \leq \bar{\delta}$ then $\mathrm{diam}(X) \geq \pi - \eta$.
- Step 2 conclude by a compactness-contradiction argument:
    - Assume by contradiction there exist $\varepsilon_0 > 0$ and a sequence $(X_j, \mathrm{d}_j, \mathfrak{m}_j)$ of $RCD(N-1-\frac{1}{j}, N+\frac{1}{j})$ spaces such that $\mathcal{I}_{(X_j,\mathrm{d}_j,\mathfrak{m}_j)}(v) \leq \mathcal{I}_{N-1,N,\infty}(v) + \frac{1}{j}$ but $\mathrm{d}_{mGH}(X_j, [0,\pi] \times_{\sin}^{N-1} Y) \geq \varepsilon_0$ for every $j \in \mathbb{N}$ and every $RCD(N-2, N-1)$ space $(Y, \mathrm{d}_Y, \mathfrak{m}_Y)$ with $\mathfrak{m}_Y(Y) = 1$.
    - Then by Step 1 we get $\mathrm{diam}(X_j) \to \pi$
    - by Gromov's compactness Theorem + stability of $RCD(N-1, N)$ there exists an $RCD(N-1, N)$ space $(X_\infty, \mathrm{d}_\infty, \mathfrak{m}_\infty)$ such that, up to subsequences, $X_j \to X_\infty$ in $mGH$-sense.
    - by since diam is continuous under $mGH$-convergence, we get $\mathrm{diam}(X_\infty) = \pi$, so by Max Diam Thm $X_\infty$ is a spherical suspension; contradiction.

# Further results via 1-D localization

In a second paper still in collaboration with Cavalletti we used 1-D localization to prove further inequalities, many of them answer open problems proposed by Villani in its celebrated book "Optimal transport: old and new".

If $(X, \mathrm{d}, \mathfrak{m})$ is $RCD(K, N)$ with $K > 0$, then

- *p*-spectral gap: Let $\lambda^{1,p}_{(X,\mathrm{d},\mathfrak{m})} = \inf\left\{ \frac{\int_X |\nabla f|^p d\mathfrak{m}}{\int_X |f|^p d\mathfrak{m}} : f \neq 0, \int_X f|f|^{p-2} d\mathfrak{m} = 0 \right\}$, then $\lambda^{1,p}_{(X,\mathrm{d},\mathfrak{m})} \geq \lambda^{1,p}_{K,N}$, with "=" iff $X$ is a spherical suspension, and "almost =" iff $X$ is $mGH$-close to a spherical suspension.
- Dimensional improvement of Log-Sobolev
  for any $f : X \to [0, \infty)$ with $\int_X f \; d\mathfrak{m} = 1$ it holds
  $2 \frac{KN}{N-1} \int_X f \log f \; d\mathfrak{m} \leq \int_{\{f>0\}} \frac{|\nabla f|^2}{f} d\mathfrak{m}$,
- Sharp Sobolev
  $\frac{KN}{(p-2)(N-1)} \left\{ \left(\int_X |f|^p \, d\mathfrak{m}\right)^{\frac{2}{p}} - \int_X |f|^2 \, d\mathfrak{m} \right\} \leq \int_X |\nabla f|^2 \, d\mathfrak{m}$,

## Euclidean tangents to $RCD(K,N)$ spaces

- Cheeger-Colding '97: for limit spaces the local blow ups are a.e. unique and euclidean.
- Q: is it true also for $RCD(K,N)$ spaces?
- Notation Fixed $\bar{x} \in X$, call $Tan(X, \mathrm{d}, \mathfrak{m}, \bar{x})$ the set of local blow ups (also called tangent cones) of $X$ at $\bar{x}$.
- More precisely, for $r \in (0,1)$ consider the p.m.m.s. $(X, r^{-1}\mathrm{d}, \mathfrak{m}(B_r(\bar{x}))^{-1} \cdot \mathfrak{m}, \bar{x})$.
  Given any sequence $r_n \downarrow 0$, by Gromov compactness, there exists a subsequence $r_{n_k} \downarrow 0$ and a limit space $(Y, \mathrm{d}_Y, \mathfrak{m}_Y, \bar{y})$ such that $(X, r_{n_k}^{-1}\mathrm{d}, \mathfrak{m}(B_{r_{n_k}}(\bar{x}))^{-1} \cdot \mathfrak{m}, \bar{x}) \to (Y, \mathrm{d}_Y, \mathfrak{m}_Y, \bar{y})$.
  By definition $Tan(X, \mathrm{d}, \mathfrak{m}, \bar{x})$ is the set of all these limit spaces $(Y, \mathrm{d}_Y, \mathfrak{m}_Y, \bar{y})$.

# Existence of Euclidean tangents

THM 1 [Gigli-M.-Rajala '13] Let $(X, \mathsf{d}, \mathfrak{m})$ be an $RCD(K, N)$ space. Then for $\mathfrak{m}$-a.e. $x \in X$ there exists $n = n(x) \in \mathbb{N}$, $n \leq N$, such that

$$(\mathbb{R}^n, \mathsf{d}_E, \mathcal{L}_n, 0) \in \mathrm{Tan}(X, \mathsf{d}, \mathfrak{m}, x),$$

where $\mathsf{d}_E$ is the Euclidean distance and $\mathcal{L}_n$ is the $n$-dimensional Lebesgue measure normalized so that $\int_{B_1(0)} 1 - |x| \, d\mathcal{L}_n(x) = 1$.

## Idea of proof

The key technical tool of the proof is the Splitting theorem in $RCD(0, N)$ spaces by Gigli (non smooth generalization of the classical Cheeger-Gromoll Splitting Thm)

1. $\mathfrak{m}$-a.e. $\bar{x} \in X$ is the midpoint of some geodesic
2. Take a sequence of blow ups at such $\bar{x}$, by Gromov compactness and by Stability they converge to a limit $RCD(0, N)$ space $(Y, \mathsf{d}_Y, \mathfrak{m}_Y, \bar{y}) \in \mathrm{Tan}(X, \mathsf{d}, \mathfrak{m}, \bar{x})$
3. By the choice of $\bar{x}$, $Y$ contains a line and therefore splits an $\mathbb{R}$ factor, by the splitting thm: $Y \cong Y' \times \mathbb{R}$
4. Repeating the construction for $Y'$ in place of $X$ we get that there exists a local blow up $\tilde{Y}'$ of $Y'$ that splits an $\mathbb{R}$ factor: $\tilde{Y}' = Y'' \times \mathbb{R}$
5. Adapting ideas of Preiss (and of Le Donne) we prove that $\mathfrak{m}$-a.e. tangents of tangents are tangent themselves, i.e. $Y'' \times \mathbb{R}^2 = \tilde{Y}' \times \mathbb{R} \in Tan(X, \mathsf{d}, \mathfrak{m}, \bar{x})$
6. repeating the scheme iteratively we conclude.

## Further structure of $RCD(K,N)$ spaces

Q: In the previous Thm we have existence of a euclidean tangent cone; but is the tangent cone unique?

THM 2[Naber-M.'14] Let $(X, \mathsf{d}, \mathfrak{m})$ be an $RCD(K,N)$ space. Then for $\mathfrak{m}$-a.e. $x \in X$ the tangent cone IS UNIQUE and euclidean, i.e. there exists $n = n(x) \in \mathbb{N}$, $n \leq N$, such that

$$\{(\mathbb{R}^n, \mathsf{d}_E, \mathcal{L}_n, 0)\} = \mathrm{Tan}(X, \mathsf{d}, \mathfrak{m}, x),$$

More precisely we have
THM 3[Naber-M.'14] [Rectifiability of $RCD(K,N)$-spaces] Let $(X, \mathsf{d}, \mathfrak{m})$ be an $RCD(K,N)$ space. Then, for every $\varepsilon > 0$ there exists a countable collection $\{R_j^\varepsilon\}_{j\in\mathbb{N}}$ of $\mathfrak{m}$-measurable subsets of $X$, covering $X$ up to an $\mathfrak{m}$-negligible set, such that each $R_j^\varepsilon$ is $1+\varepsilon$-biLipshitz to a measurable subset of $\mathbb{R}^{k_j}$, for some $1 \leq k_j \leq N$, $k_j$ possibly depending on $j$.

# Preliminary remarks

- If $X$ is a Ricci limit space, Thm 2 was first proved by Cheeger-Colding '00: prove hessian estimates on harmonic approximations of distance functions, and use these to force splitting behavior.
- In the context of general metric spaces the notion of a hessian is still not at the same level as it is for a smooth manifold, and cannot be used in such strength (interesting work of Gigli in this direction though).
- So we prove entirely new estimates, both in the form of gradient estimates on the excess function and a new almost splitting theorem with excess, which will allow us to use the distance functions directly as our chart maps. New even in the smooth context.

## Strategy of proof, 1: the $A_k$'s.

Define

$A_k := \{x \in X \; : \; \exists$ a tangent cone of X at x equal to $\mathbb{R}^k$ but no tangent cone at x splits $\mathbb{R}^{k+1}\}$.

We first prove that

-$A_k$ is $\mathfrak{m}$-measurable (it is difference of analytic sets),

- by THM 1 we get $\mathfrak{m}(X \setminus \bigcup_{k \in \mathbb{N}} A_k) = 0$.

So THM 2-3 are a consequence of the following

THM 4. Let $(X, \mathsf{d}, \mathfrak{m})$ be an $RCD(K, N)$-space, and let $A_k$ be as above. Then

(1) For $\mathfrak{m}$-a.e. $x \in A_k$ the tangent cone of $X$ at $x$ is unique and isomorphic to the $k$-dimensional euclidean space.

(2) There exists $\bar{\varepsilon} = \bar{\varepsilon}(K, N) > 0$ such that, for every $0 < \varepsilon \leq \bar{\varepsilon}$, $A_k$ is $k$-rectifiable via $1 + \varepsilon$-biLipschitz maps. More precisely, for each $\varepsilon > 0$ we can cover $A_k$, up to an $\mathfrak{m}$-negligible subset, by a countable collection of sets $U^k_\varepsilon$ with the property that each one is $1 + \varepsilon$-biLipschitz to a subset of $\mathbb{R}^k$.

# Strategy of proof, 2: rough idea

1. Given $\bar{x} \in A_k$, for every $0 < \delta << 1$ there exists $r > 0$ such that $\mathrm{d}_{mGH}(B_{\delta^{-1}r}(\bar{x}), (B_{\delta^{-1}r}(0^k)) \leq \delta r$.
2. For some radius $r << R << \delta^{-1} r$ we can then pick points $\{p_i, q_i\}_{i=1,\dots,k} \in X$ corresponding to the bases $\pm Re_i$ of $\mathbb{R}^k$. Define the map $\vec{d} = \Big(\mathrm{d}(p_1,\cdot) - \mathrm{d}(p_1,\bar{x}), \dots, \mathrm{d}(p_k,\cdot) - \mathrm{d}(p_k,\bar{x})\Big) : B_r(\bar{x}) \to \mathbb{R}^k$. For $\delta$ sufficiently small, $\vec{d}$ is a $\varepsilon r$-mGH map $B_r(\bar{x}) \to B_r(0^k)$.
3. MAIN CLAIM: $\exists$ a set $U_\varepsilon \subseteq B_r(\bar{x})$ of almost full measure, i.e. $\mathfrak{m}(B_r(\bar{x}) \setminus U_\varepsilon) \leq \varepsilon$, s.t. $\forall x \in U_\varepsilon$ and $s \leq r$, the restriction map $\vec{d} : B_s(x) \to \mathbb{R}^k$ is an $\varepsilon s$-measured Gromov Hausdorff map.
4. From this we can show that the restriction map $\vec{d} : U_\varepsilon \to \mathbb{R}^k$ is in fact $1+\varepsilon$-bilipschitz onto its image. By covering $A_k$ with such sets we will show that $A_k$ is itself rectifiable.

## Strategy of proof, 3: two new ingredients

Define $e_{p,q}(y) := \mathsf{d}(p,y) + \mathsf{d}(q,y) - \mathsf{d}(p,q)$, called excess function. In order to get the main claim, two new ingredients

1. Gradient Excess Estimates. We show that the gradient of the excess functions $e_{p_i,q_i}$ of the points $\{p_i, q_i\}$ is small in $L^2$, more precisely: for the above $\delta > 0$ small enough, then
$$\fint_{B_r(\bar{x})} |De_{p_i,q_i}|^2 \, d\mathfrak{m} \le \varepsilon_1.$$
2. Almost splitting via excess: given $x \in B_r(\bar{x})$ and $s \in (0,r)$, if $\fint_{B_s(x)} |De_{p_i,q_i}|^2 \, d\mathfrak{m} < \varepsilon_1$, then
$$\mathsf{d}_{mGH}\left(B_s(x), B_s^{\mathbb{R}\times Y}((0,y))\right) < \varepsilon_2 \, s,$$
for some m.m.s. $(Y, \mathsf{d}_Y, \mathfrak{m}_Y, y)$.
I.e.: gradient of excess small in $L^2$ $\Rightarrow$ close to a splitting.
Proof by contradiction, in the limit we enter into the framework of the arguments of Splitting Theorem.

# Strategy of proof, 4: construction of $U_\varepsilon$

Construction via a maximal function argument: for $x \in B_r(\bar{x})$ call

$$M(x) := \sup_{s\in(0,r)} \sum_{i=1}^{k} \fint_{B_s(x)} |De_{p_i,q_i}|^2 \, d\mathfrak{m}.$$

Define

$$U_\varepsilon := \{x \in B_r(\bar{x}) : M(x) < \varepsilon\}.$$

By the Gradient Excess Estimates+ $L^1 \to L^{1,weak}$ continuity of maximal function operator
$\Rightarrow$ for $\delta > 0$ small enough we have $\mathfrak{m}(B_r(\bar{x}) \setminus U_\varepsilon) < \varepsilon$.

But $\forall x \in U_\varepsilon$, $\forall s \leq r$, by construction,
$\sum_{i=1}^{k} \fint_{B_s(x)} |De_{p_i,q_i}|^2 \, d\mathfrak{m} \leq \varepsilon \, s$. An iteration of the almost splitting theorem via excess estimates implies then that

$$\mathrm{d}_{mGH}(B_s(x), B_s(0^k)) \leq \varepsilon_2 \, s, \quad \forall s \leq r \quad \Rightarrow \quad \text{Main claim.}$$

## Challenges for the future

- As Alexandrov spaces played a crucial role to establish new theorem for smooth manifolds with lower sectional curvature bounds, we expect $RCD(K, N)$ spaces to be useful to give new insights for smooth manifolds with lower Ricci bounds.
- For Ricci limits, Colding-Naber '11 and Kapovitch-Li '15 proved that the dimension of the euclidean tangent space is constant a.e. Is it true also also for $RCD(K, N)$ spaces?
- Is it true that any $RCD(K, N)$ space is a Ricci limit?
- Is it true that any $RCD(1, 2)$ space is Alexandrov with curv$\geq 1$?
- "Ricci flow" for metric measure spaces? Some interesting recent insights by Haslhofer-Naber, Kleiner-Lott, Lott, Gigli-Mantegazza, Sturm,...

!!THANK YOU FOR THE ATTENTION!!