Smoothing Properties and Uniqueness of the Weak Kähler-Ricci Flow

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Today's talk will be about joint work with Chih Lu (Centro de Giorgi Pisa). From now on, we let X be a compact Kähler manifold of complex dimension n and $\alpha_0 \in H^{1,1}(X, \mathbb{R})$. We call a Kähler form an element $\omega \in \alpha_0$ which is a real, closed, positive $(1,1)$-form. We say that the family of Kähler metrics (ω_t) solves the Kähler Ricci flow starting from ω_0 if it satisfies the following parabolic equation:

$$\begin{aligned} \frac{\partial \omega_t}{\partial t} &= -Ric(\omega_t) \\ \omega_t|_{t=0} &= \omega_0. \end{aligned}$$

Recall that locally we can express the metric ω as

$$\omega \sim i \sum_{\alpha,\beta} \omega_{\alpha\overline{\beta}} dz_\alpha \wedge d\overline{z}_\beta$$

and the Ricci form of ω is given by

$$Ric(\omega) \sim -\frac{i}{\pi} \sum_{\alpha,\beta} \frac{\partial^2}{\partial z_\alpha \partial \overline{z}_\beta} \log \det(\omega_{\alpha\overline{\beta}}) dz_\alpha \wedge d\overline{z}_\beta$$

so that $\{Ric(\omega)\} = c_1(X) := -c_1(K_X)$.

The Kähler-Ricci flow became one of the major tool in Kähler geometry starting form the work of Cao '85: he proved that when $c_1(X) \leq 0$, the flow converges to the Kähler-Einstein metric enodwed by the manifold, i.e. $\omega_t \to \omega_{KE}$ as $t \to \infty$. The study of the Kähler-Ricci flow is also motivated by the analytic analogue of the Minimal Model Program proposed by Song-Tian '07. This is a ver ambitious project but it would give the classification not only of projective manifolds but also of Kähler manifolds. In order to do that one has to (re)start the flow from a degenerate initial data, namely from a closed positive (1,1)-current. One can think of a **current** as the dual of differential forms. A positive current corresponds to differential forms with measures as coefficients. Examples include Kähler forms and current of integration along a divisor.

The following results are currently known about the Kähler-Ricci flow:

1. Cao '85, Tsuji '88, Tian-Zhang '06: if ω_0 is a Kähler metric then there exists a unique family of Kähler metrics $(\omega_t)_{[0,T_{\max})}$ solving the Kähler-Ricci flow with $\omega_t|_{t=0} = \omega_0$. Moreover,

$$T_{\max} = \sup\{t > 0 \, : \, \alpha_0 - t c_1(X) > 0\}.$$

2. Song-Tian in '09 showed that when we can run the flow from a positive current with continuous potential, i.e. $T_0 \simeq i\partial\overline{\partial} u_0$

   where u_0 is a continuous function, then there exists a unique family $(\omega_t)_{t\in(0,T_{\max})}$ of Kähler metrics solving the Kähler-Ricci flow and $\omega_t \to T_0$ as $t \to 0^+$ unformly (i.e., in L^∞ at the level of potentials).

3. Guedj-Zeriahi '13 showed that if T_0 is a positive $(1,1)$-current with zero Lelong numbers then there exists a family $(\omega_t)_{t\in(0,T_{\max})}$ of Kähler metrics solving the Kähler-Ricci flow, and $\omega_t \to T_0$ as $t \to 0^+$ in L^1 in the weak sense (i.e. in L^1 at the level of potentials).

Now, what happens in the case of positive Lelong numbers?

Theorem 1 (Di Nezza-Lu) *Fix $\varepsilon > 0$. Given a positive closed $(1,1)$-current T_0 with a technical condition (to be described later), then there exists a family $(\omega_t)_{t\in(\epsilon,T_{\max})}$ of positive closed $(1,1)$-currents, and a closed analytic subset D_ϵ such that*

1. *ω_t is smooth on $X \setminus D_\epsilon$ and there ω_t solves the Kähler-Ricci flow in the classical sense;*
2. *$\omega_t \to T_0$ weakly as $t \to 0^+$.*

Comments: if we fix a Kähler form $\omega \in \alpha_0$, then any positive closed $(1,1)$-current writes as $T_0 = \omega + dd^c\varphi_0$, where φ_0 is a **ω-plurisubharmonic** function, meaning that locally φ_0 writes as sum of a smooth function and a psh function, and $\omega + dd^c\varphi_0 \geq 0$ in the weak sense.
For any $x \in X$, we define the Lelong number $\nu(T_0, x)$ by

$$\nu(T_0,x) := \nu(\varphi_0,x) = \sup\{\gamma > 0 \mid \varphi_0(z) \leq \gamma \log|z-x| + C\}.$$

For a Kähler form ω we have that $\nu(\omega, x) = 0$ for any point $x \in X$. For a current of integration along a smooth divisor, $[D]$, we have that $\nu([D], x) = 0$ for $x \notin D$ and $\nu([D], x) = 1$ for $x \in D$.

The technical condition contained in the above statement says that $T_{\max} > \frac{1}{2c(\varphi_0)}$ where $c(\varphi_0)$ is the **critical index of integrability**. Note that $c(\varphi_0) = +\infty$ if and only if $\nu(\varphi_0, x) = 0$ for all $x \in X$. Additionally we have that

$$D_\epsilon = \{x \in X \mid \nu(T_0, x) \geq \epsilon\}$$

where D_ϵ is closed anad analytic subset of X(result given by Siu '74).
The Kähler-Ricci flow reduces to a scalar parabolic equation. In particular, one can prove that $\omega_t = (\omega - tRic(\omega)) + dd^c\varphi_t$ solves the Kähler-Ricci flow if and only if

$$(CMAE) \qquad (\omega - tRic(\omega) + dd^c\varphi_t)^n = e^{\dot{\varphi}_t}.$$

As a result we are able to rephrase the theorem at the level of potentials:

Theorem 2 *Fix $\varepsilon > 0$ If φ_t is an ω-plurisubharmonic function (with $\nu(\varphi_0, x) > 0$ for some $x \in X$) and $T_{\max} > \frac{1}{2c(\varphi_0)}$ then there exists φ_t which is a solution of the (CMAE). Moreover, for all $t > \epsilon$ we have $\varphi_t \in C^\infty(X \setminus D_\epsilon)$ and $\varphi_t \to \varphi_0$ in L^1 as $t \to 0^+$.*

Idea of the Proof: Let φ_0 be a plurisubharmonic function and let $\varphi_{0,j}$ be a sequence of smooth ω-plurisubharmonic functions decreasing to φ_0. Tian-Zhang showed there exists a unique smooth sequence $\varphi_{t,j}$ solving the Kähler-Ricci flow srarting from $\varphi_{0,j}$. We want to establish uniform estimates for $\varphi_{t,j}$ on any K compact subset of $X \setminus D_\epsilon$:

$$||\varphi_{t,j}||_{C^{k,\alpha}(K\times(\epsilon,T_{\max}))} < C(K).$$

The most difficult estimate is the C^0 estimate.

Theorem 3 *Fix $\varepsilon > 0$ and $T < T_{\max}$ then*

$$\varphi_t \geq \left(2 - \frac{t}{2T}\right)\psi - C(\epsilon, T)$$

where ψ is a plurisubharmonic function and $\psi \in C^\infty(X \setminus D_\epsilon)$.

In order to prove such a theorem we need *pluripotential theory.*

Description of the Flow

It was shown by Guedj-Zeriahi '13 that if $t > \frac{1}{2c(\varphi_0)}$ then the flow $\varphi_t \in C^\infty(X)$.
But what happens in the short time range? The following result shows that we have a phenomenum of propagation of singularities.

Theorem 4 *We have that*

$$D_c = \{x \in X \mid \nu(\varphi_0, x) \geq c\} \subset \{x \in X \mid \nu(\varphi_t, x) \geq c - 2nt\}.$$

In particular, when $t < \frac{1}{2nc(\varphi_0)}$, $D_c \neq \varnothing$, *hence the flow* φ_t *has singularities at* $-\infty$.

Proof: From the definititon of Lelong numbers, there exists a function Φ that locally looks like $\Phi \sim c \log ||z||$ and such that $\varphi_0 \leq \Phi$. Using the maximum principle we can prove that

$$\varphi_t \leq \left(1 - \frac{2nt}{c}\right)\Phi - ct$$

hence, $\nu(\varphi_t, x) \geq \left(1 - \frac{2nt}{c}\right)\nu(\Phi, x)$ where $\nu(\Phi, x) = c$.

Uniqueness

The uniqueness in the general case of φ_0 having positive Lelong numbers is an open research question. We are guaranteed uniqueness if φ_0 has $\nu(\varphi_0, x) = 0$ for all $x \in X$.

Definition 1 *We say that* φ_t *is a* ***weak Kähler-Ricci flow*** *if*

- φ_t *is* $(\omega - tRic(\omega))$*-plurisubharmonic, for any* $t > 0$*;*
- φ_t *is smooth for all* $t > 0$ *and it solves the Kähler-Ricci flow for all* $t > 0$*;*
- $\varphi_t \to \varphi_0$ *in* L^1.

Theorem 5 *Let* φ_0, ψ_0 *be* ω*-plurisubharmonic functions such that* $\nu(\varphi_0, x) = 0$ *for all* $x \in X$*. Let* φ_t, ψ_t *be the weak flows starting from* φ_0 *and* ψ_0*, respectively. If* $\varphi_0 \leq \psi_0$ *then* $\varphi_t \leq \psi_t$ *for all* t*. In particular, the flow is unique.*

We can in some sense see this theorem as a maximum principle in the general setting.