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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Mee Seong Im **Email/Phone:** Mim2@illinois.edu

Speaker's Name: Olivier Schiffmann

Talk Title: Quivers, curves, Kac polynomials and the number of stable Higgs bundles

Date: 9 / 3 / 14 **Time:** 3 : 30 am / (pm) (circle one)

List 6-12 key words for the talk: quivers, Kac polynomials, indecomposable vector bundles, stable Higgs bundles

Please summarize the lecture in 5 or fewer sentences: Schiffmann replaces the category of representations of a quiver by the category of coherent sheaves on a smooth proj. curve, explaining a "global" analog of a certain conjecture by Kac. As an application, Schiffmann gives a formula for the # of stable Higgs bundles over a sm proj. curve over a finite field.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

☑ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.

☑ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.

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Quivers, curves, Kac Polynomials and the number of stable Higgs bundles

Olivier Schiffmann

Wed, Sept 3, 2014, 3:30–4:30 pm

Plan: today count indecom. v. bdles on curves (no geometry)

next: relate it to Higgs st (geometric, symplectic algebraic (Mozgovey–S)

§ 3. Spherical Hall algebras of curves

(Kapranov '97)

X sm, proj. curve $/\mathbb{F}_q$

$$\zeta_X(z) = \frac{\prod(1-\alpha_i z)}{(1-z)(1-qz)}$$

$Coh_{r,d}$ = cat. of coh. sheaves on X of class (r,d)

$$H_{r,d} := \{ f : Coh_{r,d} \to \mathbb{C} : \# \operatorname{Supp} f < \infty \}$$

$$H_X = \bigoplus_{r,d} H_{r,d}$$

$v = q^{-1/2}$

$\underline{m}$: $H_{r,d} \otimes H_{r',d'} \longrightarrow H_{r+r', d+d'}$

$$f \otimes g \longmapsto \left(\mathcal{F} \mapsto v^{-\langle \mathcal{F}/\mathcal{H}, \mathcal{H} \rangle} \sum_{\mathcal{H} \subseteq \mathcal{F}} g(\mathcal{H}) f(\mathcal{F}/\mathcal{H}) \right)$$

endows H_X with assoc. alg. structure.

$\underline{\Delta}$: $H_{r+r', d+d'} \longrightarrow H_{r,d} \hat{\otimes} H_{r',d'}$

$$h \longmapsto \left((\mathcal{F}, \mathcal{G}) \mapsto \frac{v^{-\langle \mathcal{F}, \mathcal{G} \rangle}}{\# \mathrm{Ext}^1(\mathcal{F}, \mathcal{G})} \sum_{\xi \in \mathrm{Ext}^1(\mathcal{F}, \mathcal{G})} h(X_\xi) \right)$$

Green pairing

$$(f|g) = \sum_{\mathcal{F}} \frac{f(\mathcal{F}) g(\mathcal{F})}{\# \mathrm{Aut}^0(\mathcal{F})} = \int fg$$

<u>Thm</u>: (Ringel, Green)

$(H_X; m, \Delta, (\,,\,))$ is a self-dual (topological, twist) Hopf algebra.

i.e., $(fg|h) = (f \otimes g | \Delta(h))$

<u>Def</u> H_X^{sph} = subalg. gen. by $\{ 1_{\mathrm{Pic}^d} : d \in \mathbb{Z} \}$ & $\{ 1_{\mathrm{Coh}_{0,d}} : d \in \mathbb{N} \}$.

Prop i.) H_X^{sph} is a self-dual Hopf algebra.
ii.) $CT_{H^{sph}_{X,r}} = \Delta_{1,\dots,1} : H^{sph}_{X,r} \mapsto H^{sph\,\otimes r}_{X,1}$
is injective.

Rmks. i.) $H_X^{Bun} = \{ f \in H_X : supp(f) \subseteq Bun \}$

H_X ↑ $\bigoplus_r$ A.F. for GL_r

$H_X^{Tor} = \bigoplus_d H_{X,0,d}$

↑ algebra of Hecke operators.

$$H_X = H_X^{Bun} \ltimes H_X^{tor}$$

ii.) Via Faisceaux-Function correspondence, & up to a completion, H_X^{sph} corresponds to $\bigoplus K_0(Eis_{GL_r})$.

Prop. i.) $\forall\ \underline{\alpha} = (\alpha_1, \dots \alpha_r)$, $\mu(\alpha_1) < \dots < \mu(\alpha_r)$.
$1_{HN_{\underline{\alpha}}} \in H_X^{sph}$

ii.) T_g, W_g, $R_g = \mathbb{Q}[T_g]^{W_g} =$

$= \mathbb{Q}[z_1, \dots z_r : z_{2i-1} z_{2i} = z_{2j-1} z_{2j}]^{W_g}$

$\exists$ a R_g-Hopf alg. ${}_R H^{sph}$ containing elmts ${}_R 1_{HN_{\underline{\alpha}}}$ s.t. $\forall \mathbb{F}_q$, $\forall X$

$$H_X^{sph} \simeq {}_R H^{sph} \underset{\tau_i \to \alpha_i}{\otimes} \mathbb{C}$$

☆ Eisenstein series

Def $E_r(z) = \sum_{d\in\mathbb{Z}} 1_{Coh_{r,d}} (v^{r-1} z)^d \in \prod_{d\in\mathbb{Z}} \hat{H}_X$

$$E_r^{Bun}(z) = \sum 1_{Bun_{r,d}} \overline{}$$

$$E_{r_1}(z_1) \cdots E_{r_s}(z_s) \in \prod_d \hat{H}_X^{sph}[[z_1^{\pm 1}, \ldots, z_s^{\pm 1}]]$$

Thm. (Harder)

$\forall \mathcal{F} \in Coh_{r,d}$, $E_{r_1}(z_1) \cdots E_{r_s}(z_s)(\mathcal{F})$ is ~~the~~ expansion in $z_1 \ll z_2 \ll \cdots \ll z_s$ of a rat'l fn.

" Functional Equat. "

- $E_r(z) = E_r^{Bun}(z)\, E_0(v^{2r} z)$
- $\Delta(E_r(z)) = \sum_{r_1+r_2=r} v^{(1-g)r_1 r_2} E_{r_1}(z) \otimes E_{r_2}(v^{2r_1} z)$
- $E_0(z)\, E_r(w) = \prod_{i=0}^{r-1} \varsigma\left(z^{-2i} \frac{z}{w}\right) E_r(w)\, E_0(z)$
- $E_1(z)\, E_1(w) = \left(\frac{z}{w}\right)^{2(g-1)} E_1(w)\, E_1(z)$

Set $\tilde{\xi}_X(q^{-1}) = \dfrac{\prod (1-\alpha_i)}{1-q^{-1}}$

Thm (Harder)

$$\operatorname{Res}_{\frac{z_1}{z_2} = \cdots = \frac{z_{r-1}}{z_r} = q^{-1}} E_1^{Bun}(z_1) \cdots E_1^{Bun}(z_r) = \frac{v^{-}\tilde{\xi}_X(q^{-1})^{r-1}}{c\{\xi_X(q^{-2}) \cdots \xi_X(q^{-r})} E_r^{Bun}(z_r)$$

Sketch of proof

$CT(\operatorname{Res}_{-} E_1(z_1) \cdots E_1(z_r))$

$CT(E_r^{Bun}(z_r))$

\+ CT injective on H_X^{sph}

Sketch of proof of Siegel's formula:

$$\operatorname{vol}(Bun_{r,d}) = (1_{Bun_{r,d}} \mid 1_{Bun_{r,d}})$$
$$= c \operatorname{Res}_{-} (E_1(z_1) \cdots E_1(z_r) \mid 1_{Bun_{r,d}})$$
$$= c \cdot \operatorname{Res}_{-} (E_1(z_1) \otimes \cdots \otimes E_1(z_r) \mid \underbrace{CT(1_{Bun_{r,d}})}_{\text{"}1_{Bun_{r,d}} \otimes \cdots \otimes 1_{Bun_r}\text{"}})$$

§. Counting indecomp. vector bdles

$Coh(X)$ is a Krull-Schmitt category

Def $\mathcal{V} \in Coh(X)$ is abs. indecom. if $\mathcal{V} \otimes \overline{\mathbb{F}}_q$ is indecomp.

Put $A_{r,d}(X)$ = # abs. indecomp v-bdles on X of $\mathcal{C}l(r,d)/\sim$

Q: $A_{r,d}(X) = ?$

Ex $r=1$

$$A_{1,d}(X) = \# \mathrm{Pic}^d = \prod (1-\alpha_i)$$

$$g=0, \quad A_{r,d} = \begin{cases} 1 & \text{if } r=1 \\ 0 & \text{if } r \neq 1 \end{cases}$$

$$g=1 \quad A_{r,d} = \# X(\mathbb{F}_q) = 1+q-\sum \alpha_i \quad \forall r,d$$

$$A_{g,2,d} =$$

$$= \prod(1-\alpha_i) \left\{ \frac{\prod(1-q\alpha_i)}{(q-1)(q^2-1)} - \frac{\prod(1+\alpha_i)}{4(1+q)} + \frac{\prod(1-\alpha_i)}{2(q-1)} \left[\frac{1}{2} - \frac{1}{q-1} - \sum \frac{1}{1-\alpha_i} \right] \right\}$$

Note: indecomp. of d.

Rmk Quivers ⇝ Kac proved existence of pols counting indec's.

$$\# \text{ abs. indecomp} = \sum_{\mathcal{F} \text{ abs. indecomp}} 1 \neq \sum_{\mathcal{F} \text{ abs. indecomp}} \frac{1}{\# \mathrm{Aut}(\mathcal{F})} = \mathrm{vol}\left(\text{Abs. indecomp}_{r,d}\right)$$

↑ Constructible substack of Bun_{GL_r}.

First idea:
count all v. bdles /~
& use recursion.

Thm: # v. bdles of $\mathcal{M}(r,d)$ is ∞!
Solution: • truncation $Coh^{\geq 0}(X)$ of $Coh(X)$
• count all bdles in $Coh^{\geq 0}(X)$

Def. $\mu_{min}(\mathcal{F}) = \mu(\alpha_1)$ if $HN(\mathcal{F}) = (\alpha_1, \dots, \alpha_s)$

$Coh^{\geq 0}(X)$ subcat. of coh. sheaves satisfying $\mu_{min} \geq 0$

Facts. $Coh^{\geq 0}$ is stable under extensions, quotients, & summands.

$A^{\geq 0}_{r,d}(X) = \#$ abs. indec. in $Coh^{\geq 0}_{r,d}/\sim$.

$A^{\geq 0}_{r,d}(X) = A_{r,d}(X)$ for $d \gg 0$.

Enough to compute $vol(Coh^{\geq 0}_{r,d}(X))$ $\forall r, d$.

$\rightsquigarrow$ considering a partition of the stack

$Nil_{r,d}(X) = \{(\mathcal{F}, \theta) : \mathcal{F} \in Coh_{r,d}, \theta \in End^{nil}(\mathcal{F})\}$

$Nil_{r,d}(X) = \coprod_{\underline{\alpha}} J_{\underline{\alpha}}$

$$\mathrm{Nil}^{\geq 0}_{r,d}(X) = \coprod_{\underline{\alpha}} J^{\geq 0}_{\underline{\alpha}}$$

↑ Compute using Eisenstein series

$$\left(E_1^{Bun}(z_1) \cdots E_1^{Bun}(z_r) \,\middle|\, \mathbb{1}_{\mathrm{Coh}^{\geq 0}_{r,d}} \right)$$

Can compute in RH^{sph} →