DECISION PROBLEMS
(FOR GROUPS AND SPACES)

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ABSTRACT. The (non)existence and complexity of algorithms has been a central theme in combinatorial and, later, geometric group theory since their inception, with low dimensional topology providing both motivation and a significant field of application. In this talk I will review some of the milestones in the development of decision problems in group theory, highlighting the geometry behind them. I shall then survey the current state of the art, with an emphasis on applications to geometry and topology and including decision problems for profinite groups.

1. BASIC URGES

- What is the OBJECT that we've been *given*?
- ... is it *trivial*, in the sense that ... ?
- ... are two objects the *same*? (In the sense that ...) [isomorphic]
- ... does this object X have property that ...
- CLASSIFY!

  i.e. make a *complete* and *irredundant* list $X_0, X_1, X_2, \dots$ so that every object X under consideration is isomorphic to exactly one member of the list. "Normal forms"

Example.
- finite groups
- compact 3-manifolds (given by a triangulation)

$\sim$ audience question on whether it really is easy to write down a list of all finite groups $\sim$

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After listing finite groups $G_1, G_2, G_3, \ldots$, we need to decide isomorphism between pairs of the same order, which is simply a finite check.

Remark. With many finitely-described structures – e.g., finite sets, finite combinatorial complexes, finitely presented groups or finitely described groups (recursively presented …) – if $X_1 \cong X_2$ then one can check and certify this by a DUMB PROCESS.

Question. Give a finite simplicial complex K, of dimension d, ask if $|K| \cong \mathbb{S}^3 = \partial\Delta^{d+1}$.

To get "YES" answer needs no theory.

- Asking whether there exists subdivisions $K^{(n)}, S^{(n)}$ such that $K^{(n)} \equiv S^{(n)}$
- dumbly enumerating all subdivisions and check (blindly) if things are identical

Similarly, we can naively find an isomorphism between finitely presented groups if they are isomorphic. Enumerate all possible maps

$$G = \langle\, x_1, \ldots, x_n \,|\, r_1, \ldots, r_m \,\rangle \xrightarrow{\phi} H = \langle\, y_1, \ldots, y_n' \,|\, s_1, \ldots, s_m' \,\rangle$$

by choosing $\phi(x_i) = word_i(y_j)$. For this to be a homomorphism, we need $\phi(r_i) =_H 1$. It looks like this needs a solution to the word problem, but we are only looking for "YES" answers, and in the free group $F(y_1, \ldots, y_n)$ we can naively search for expressions

$$\phi(r_i) = \prod_{i=1}^{N} \theta_i s_i^{\pm} \theta_i^{-1}.$$

We also enumerate all homomorphisms $\psi : H \to G$, and then see if any pair are mutually inverse.

Moral of the story: we can naively show things are the same, the hard direction for decision problems is to show that things are different or that something does not exist.

Classifying objects in the sense of being able to write down a complete irredundant list is the same as being able to solve the isomorphism problem. (Verbal argument given: "$\Rightarrow$" by naively checking your two objects in question against all objects on the list in parallel, "$\Leftarrow$"

by choosing any enumeration, and removing duplicates by checking against what has been generated so far.)

2. HISTORY – C. 1900; TIETZE, DEHN (1910, 1912)

Dehn wanted to classify knots, and showed that

$$\pi_1(\mathbb{S}^3 \setminus K) \cong \mathbb{Z} \iff K \approx 0$$

which he thought reduced a hard topological problem to an easy algebraic problem.

This leads to fundamental decision problems for groups.

$\Gamma = \langle\, a_1, \ldots, a_n \,|\, r_1, \ldots, r_m \,\rangle$

- Word Problem (WP): *given* $w \in \mathrm{Free}\{a_1, \ldots, a_n\}, w =_\Gamma 1$?
- Conjugacy Problem (CP): given w, w', decide if $\exists\, x \in \Gamma$ such that $x^{-1}wx =_\Gamma w'$
- Isomorphism Problem (IP): does there exist an algorithm that can determine if two finitely presented groups Γ, Γ' are isomorphic?

[A special instance of the isomorphism problem is the Triviality Problem: decide if $\Gamma \cong 1$.]

Remark. The algorithm has to come first, and then the input (words, groups, etc). For example, for any two finitely presented groups, there is an algorithm that decides if they are isomorphic. It is either the algorithm that always says YES or the algorithm that always says NO.

3. EXPLAIN WHY THESE PROBLEMS ARE UNSOLVABLE

Remark. Unsolvability has a concrete definition in terms of Turing machines, and it's not a philosophical question. Modulo Church–Turing thesis, it also means unsolvable for any model of computation. Moreover, this also means there is no effective process that we humans can follow to get an answer.

Recursively enumerable vs recursive sets

Definition. $S \subseteq \mathbb{N}$ is *recursively enumerable* (r.e.) if there exists a Turing machine that produces S.

S is *recursive* if S and $\neg S$ are recursively enumerable.

For example

$$S : 1, 4, 9, 16, 25$$
$$\neg S : 2, 3, 5, 6, 7, 8, 10$$

We can list all 3-manifolds $M_1, M_2, M_3, \dots$, and thus finite presentations for their fundamental groups $\pi_1 M_1, \pi_1 M_2, \pi_1 M_3, \dots$.

Then we have their smallest finite quotients: $G_1, G_2, G_3, \dots$, and we could then consider their orders, which is a perfectly valid recursively enumerable set of integers.

Proposition. *There exist recursively enumerable $S \subseteq \mathbb{N}$ that are* not *recursive.*

(Proved via diagonalization argument.)

"membership problem for S is undecidable"

$G_S = \langle\, a, b, c, d \mid a^i b a^{-i} = c^i d c^{-i}, i \in S \,\rangle$

$w_i := a^i b a^{-i} c^i d^{-1} c^{-i} =_G 1$ if and only if $i \in S$, so G_S has unsolvable word problem (for S as in the above Proposition).

This group is recursively presented, but it is not finitely presented.

The first finitely presented groups with unsolvable word problem are due to P.S. Novikov, but the real breakthrough is the following:

Theorem (Higman Embedding Theorem)**.** *Every recursively presented group can be embedded in a finitely presented group.*

Thus $G_S \hookrightarrow \Gamma_S$ finitely presented, and the unsolvability of WP in G_S gives unsolvability of WP in Γ_S.

There are various improved versions of Higman Embedding Theorem (e.g. due to Sapir, Clapham,...), and this is a way to show undecidability results for nice classes of groups.

Unsolvable WP $\rightsquigarrow$ unsolvable IP (or triviality)

Take $\Gamma = \langle a_1, \ldots, a_n \mid r_1, \ldots, r_m \rangle$ with unsolvable WP, and consider $\Gamma * \langle t \rangle$, which we can present as

$$\langle t, a_1', \ldots, a_n' \mid r_1', \ldots, r_m' \rangle$$

where $a_i' = a_i t^i$ (so all of these generators have infinite order).

A word $w \in \Gamma \rightsquigarrow w' = [w, t]$

$$\Gamma^{+w} = (\Gamma, s_0, \ldots, s_n | s_i^{-1} a_i' s_i = a')$$
$$\cong \text{Free}(s_0, \ldots, s_n) \text{ if } w =_\Gamma 1, \text{or}$$
$$\text{group with unsolvable WP if } w \neq_\Gamma 1$$

Remark (Collins–Miller)**.** Construct (Γ_n) such that each Γ_n is either 1 or has an aspherical presentation and can't decide which (each Γ_n has many more relations that generators).

Corollary. *There does not exist an algorithm that, given a f.p. group, can calculate* $H_2(\Gamma, \mathbb{Z})$*.*

Since the presentation is aspherical, we can build the presentation complex which will be a classifying space, and then since there are many more relations than generators, this will have non-trivial H_2.

Moreover, this shows that you cannot construct classifying spaces of groups.

Theorem. *If $n \geq 5$, then there does not exist an algorithm to recognize $\mathbb{S}^n$.*

(The manifolds are to be given as finite simplicial complexes, will all be PL n-manifolds.)

Use topological facts (for existence).

- Whitehead: smooth compact manifolds have (PL) triangulations.
- High dimensional Poincaré conjecture: M^n is closed, $H_*(M^n) \cong H_*(\mathbb{S}^n)$ and $\pi_1 M^n = 1$, then $M^n \approx_{homeo} \mathbb{S}^n$.
- Kervaire: If $\underline{n \geq 5}$ and Γ is f.p. group with $H_1\Gamma = H_2\Gamma = 0$, then there exists a homology n-sphere with $\pi_1 M = \Gamma$.

Start by writing down n-dimensional simplicial complexes $K_1, K_2, K_3, \dots$, and we can arrange that $|K_i|$ are manifolds and moreover $H_1 K_i = H_2 K_i = 0$. Every H_*-sphere is on this list (with redundancies).

Now we have fundamental groups $\Gamma_1, \Gamma_2, \Gamma_3, \dots$, with $H_1\Gamma_i = H_2\Gamma_i = 0$, and we can't decide which $\Gamma_i \cong 1$ (via some homological algebra).

In dimension 4, the Poincaré conjecture still holds but Kervaire's result doesn't hold. However, we do have the following

Kervaire: if $H_1\Gamma = H_2\Gamma = 0$ and Γ has a balanced presentation, then $\Gamma = \pi_1(\text{homology 4-sphere})$.

Unsolved group theoretic problem: can you decide triviality for groups with a balanced presentation?

Unsolved topological problem: can you recognize the 4-sphere?

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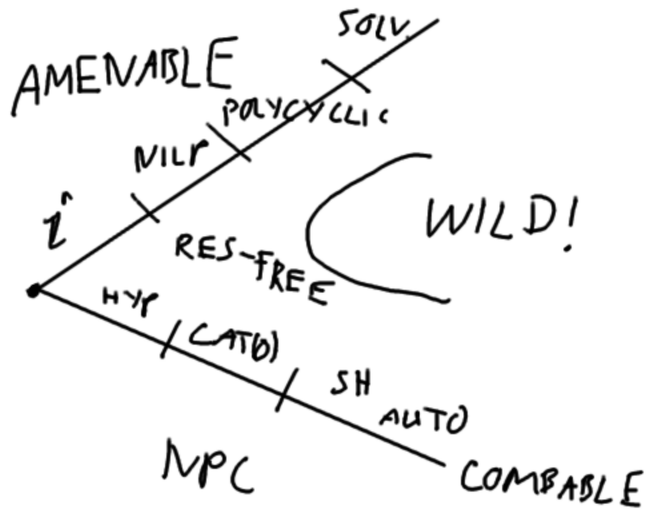

1) Balanced Presentations: Require $H_1\Gamma = 0$ (otherwise can add in extra generators)

$$\Gamma = \langle\, a_1, \dots, a_n \mid r_1, \dots, r_n \,\rangle$$

WP, CP, IP (in particular, triviality problem) are open.

1)′ Does there exist an algorithm to decide if a finite simplicial 2-complex is contractible? This is equivalent to the triviality problem for balanced presentations.

1)″ 4-sphere recognition

2) $\mathrm{Out}(F_n)$ – conjugacy problem

3) $\mathrm{SL}(3,\mathbb{Z})$ – every question about subgroups (an old favourite problem of Serre: is every finitely generated subgroup finitely presented?)

4) Hyperbolic groups

Does there exist an algorithm to decide if $\hat{\Gamma} = 1$? By work of Bridson–Wilton, this is equivalent to whether hyperbolic groups are residually finite.

Do finitely presented subgroups of hyperbolic groups have

- polynomial Dehn function (definition below)?
- solvable CP?

Definition. If $w = 1 \in \Gamma = \langle a_1, \ldots, a_n \mid r_1, \ldots, r_m \rangle$, then

$$w =_{\text{Free}} \prod_{i=1}^{N} \theta_i^{-1} r_{j(i)}^{\pm 1} \theta_i$$

and $Area(w) =$ least such N. The *Dehn function* is

$$\delta_\Gamma(n) = \max \{ Area(w) \, : \, |w| \le n, w =_\Gamma 1 \} .$$

The Dehn function is related to isoperimetric functions of manifolds.

Is the isomorphism problem for finitely presented subgroups of a fixed hyperbolic group unsolvable?

(c.f. (B, Miller) – there exists hyperbolic $\Gamma = F \rtimes F$ such that isomorphism problem for finitely presented subgroups of $\Gamma \times \Gamma \times \Gamma$ is unsolvable. You can even make Γ cubulated.)

5) Cube complexes

Is virtual specialness decidable?

Is the isomorphism problem for $\pi_1 X$ decidable?

6) Isomorphism problem for CAT(0) groups more generally (strongly suspect it to be unsolvable).

6)′ Homeomorphism problem for compact NPC manifolds (Farrell–Jones reduced this to isomorphism for fundamental groups).

Some problem stated but not written down

- Isomorphism problem for finitely-presented residually free groups.
- Can you solve IP for Kähler groups (fundamental groups of compact Kähler manifolds), and can you construct one with unsolvable WP?
- Does having a quadratic Dehn function imply solvability of the CP? Does asphericity help?
- Does every one-relator group have WP solvable in polynomial time?