Day 3 Talk 1

Artem Zvavitch

"Bezout Inequality for Mixed Volumes"

Joint w Saroglou, Soprunov

<u>Notation</u> (more on slides)

- Discuss only convex sets (bodies)

$V_n(K)$ volume of $K \in \mathbb{R}^n$

$K+L = \{k+l \mid k \in K, l \in L\}$ Minkowski sum

$V_n(\sum \lambda_i K_i)$ is a homog. pol. of degree n

$$= \sum_{i_1 \dots i_r = 1}^{r} V(K_{i_1} \dots K_{i_n}) \lambda_{i_1} \dots \lambda_{i_n}$$

$\underbrace{V(K_{i_1} \dots K_{i_n})}_{\text{coeffs}}$ ⇝ mixed volume

⊛ Mixed volume multi-linear, monotone in components

<u>Questions</u>

Q1: $D \subset \mathbb{R}^n$ convex

if $V(K_1 \dots K_r, \underbrace{D \dots D}_{n-r}) V_n(D)^{r-1} \le \prod V(K_i, \underbrace{D \dots D}_{n-1})$

$\forall$ convex bodies $K_1 \dots K_r$. Is D then a simplex?

Q2. What is the best constant $C_{n,r}$

$$V(K_1 \dots K_r, \underbrace{D \dots D}_{n-r}) V_n(D)^{r-1} \le C_{n,r} \prod V(K_i, \underbrace{D \dots D}_{n-1})$$

true for $\forall$ convex K_i, D.

<u>Plan:</u>
- How to derive?
- Why interesting?
- What is known about Q1, Q2?

Motivation (where Q1 comes from)

Bézout's theorem

Let $X_1 \ldots X_n \subset \mathbb{C}^n$ hypersurfaces, F_i polynomials

$$X_i = \{x \in \mathbb{C}^n \mid F_i(x) = 0\}. \quad \text{(Alg. variety)}$$

Assume $F_1 \ldots F_n$ are polynomials without common factors, then

$$\#(X_1 \cap \ldots \cap X_n) \leq \prod_{i=1}^{n} \deg(F_i)$$

(example on slides:  intersection of two ellipses)

Newton Polytope

Bernstein-Kushnirenko-Khovanskii:

$$\#(X_1 \cap \cdots \cap X_n) = n!\,V(P_1 \ldots P_n)$$

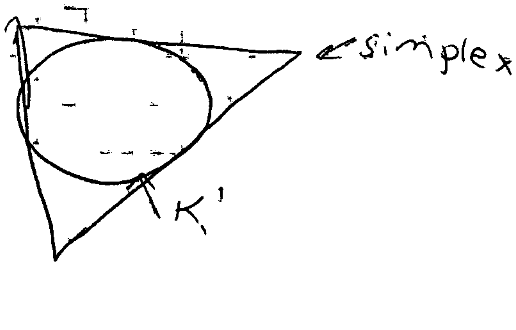
⇝ Degree Formula

$$\deg(F_i) = n!\,V(P_i, \underbrace{\Delta}_{n-1 \text{ copies of simplex}}(n-1))$$

desired inequality

thm Soprunov, A. Z '16: $V(K_1 \ldots K_r, \underbrace{\Delta \ldots \Delta}_{n-r \text{ simplices}})\, V_n(\Delta)^{r-1} \leq \prod_{i=1}^{r} V(K_i, \underbrace{\Delta \ldots \Delta}_{n-1})$

Proof sketch: by linearity assume K_i inscribed in Δ



Then

$$V(K_i, \underbrace{\Delta, \ldots, \Delta}_{n-1}) = \frac{1}{n}\sum_{v} h_{K_i}(v)\, V_{n-1}(\Delta^v)$$

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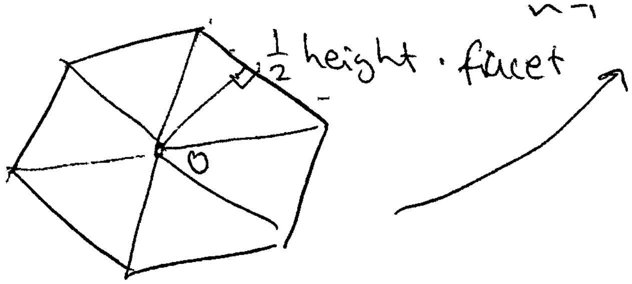
$$= V_n(\Delta)$$



Note: idea for inequality comes from Algebraic geometry, but proof by simple convex geometry.

On
<u>Q1</u>

$r=2$ hardest case ⇝ implies others

<u>Thm</u> Soprunov, A.Z '16 — use: if D is indecomposable
⇝ $D = \Delta$ ← simplex

resolves for $\mathbb{R}^2$, but not in higher dimensions

<u>Thm</u> Saroglou, Soprunov & A.Z '16

If D is a polytope ⇝ $D = \Delta$
idea of proof: perturb faces

<u>Thm</u> Saroglou, Soprunov, A.Z. '16

D no strict boundary points.

<u>Thm</u> Saroglou, Soprunov, A.Z $r = n-1$, but not known always when $r=2$

⇝ $\mathbb{R}^3$ ok $n-1=2$.

Thm Saroglou, A.Z : if K_i are zonoids

$\rightsquigarrow$

$$V(K_1, \dots K_r, D^{n-r}) V_n(D)^{r-1} \le \frac{r^r}{r!} \prod_{i=1}^{r} V(K_i, D[n-1])$$

inequality is sharp.

$\longrightarrow$

Thm if K_i are symmetric

$$V(K_1 \dots K_r, D^{n-r}) V_n(D)^{r-1} \le c_{n,r} \prod_{i=1}^{r} V(K_i, D^{n-1})$$

$c_{n,r} \le n^r r^r r!$

Is this the best?

Question 2 resolved in $n=2$ already earlier.

General case $c_{n,r} \ge \frac{r^r}{r!}$

- Artstein, Florentin, Ostrover '14 $c_{2,2} = 2$
- Brazitikos, Giannopoulos, Liakopoulos, '17 : $c_{n,2} = 2$
- —— " —— '17 $c_{n,r} \le 2^{2^{r-1}} - 1$
- Xiao '17 $c_{n,r} \le n^{r-1}$

Bezout Inequality for Mixed volumes.

Artem Zvavitch
Kent State University

(based on joint works with Christos Saroglou and Ivan Soprunov)

"Geometric functional analysis and applications"
MSRI, November 13–17, 2017.

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- We will often use notion of Minkowski sum: $K + L = \{x + y : x \in K \text{ and } y \in L\}$.
- We all know that $V_n(\lambda K) = \lambda^n V_n(K)$ for $\lambda \geq 0$, i.e. volume is a homogeneous measure of degree of homogeneity n. But there is much more!!!

Main Definitions: Mixed Volume

$K_1, K_2, \dots, K_r$ be convex bodies in $\mathbb{R}^n$ and $\lambda_1, \dots, \lambda_r \geq 0$

Then $V_n(\lambda_1 K_1 + \lambda_2 K_2 + \dots + \lambda_r K_r)$ is a homogeneous polynomial (in $\lambda_1, \dots, \lambda_r$) of degree n and

$$V_n(\lambda_1 K_1 + \lambda_2 K_2 + \dots + \lambda_r K_r) = \sum_{i_1, i_2, \dots, i_r = 1}^{r} V(K_{i_1}, \dots, K_{i_n}) \lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_n}.$$

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- Mixed volume is translation invariant: $V(K + a, K_2, \dots K_n) = V(K, K_2, \dots, K_n)$, for $a \in \mathbb{R}^n$.
- If $K \subset L$, then $V(K, K_2, K_3, \dots, K_n) \leq V(L, K_2, K_3, \dots, K_n)$.

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- Alexandrov–Fenchel inequality: $V(K_1, K_2, K_3, \dots, K_n) \geq \sqrt{V(K_1, K_1, K_3, \dots, K_n) V(K_2, K_2, K_3, \dots, K_n)}$.

Main Topic (for those who do not care about introduction)

Question 1

Fix an integer $2 \leq r \leq n$ and let $D \subset \mathbb{R}^n$ be a convex body which satisfies

$$V(K_1, \ldots, K_r, D[n-r]) V_n(D)^{r-1} \leq \prod_{i=1}^{r} V(K_i, D[n-1]),$$

for all convex bodies $K_1, \ldots, K_r \subset \mathbb{R}^n$. Is it true that then D must be n-simplex?

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is true for all convex bodies $K_1, \dots, K_r$ and D in $\mathbb{R}^n$?

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- What is known about Question 1.

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Plan

- How one could come up with such inequalities & why they are (may be) interesting?
- What is known about Question 1.
- What is known about Question 2.

Motivation: Bezout's Theorem.

Let $X_1, \dots X_n \subset \mathbb{C}^n$ be hypersrfaces defined by polynomials $F_1, \dots, F_n$:

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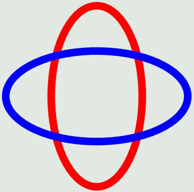
Childish Example: Two quadratic polynomials.

$$F_1(x,y) = \frac{x^2}{9} + \frac{y^2}{60} - 1 \quad \text{and} \quad F_2 = \frac{x^2}{50} + \frac{y^2}{2} - 2.$$



Then $\deg F_1 = \deg F_2 = 2$ and X_1, X_2 are ellipses which intersect in exactly 4 points.

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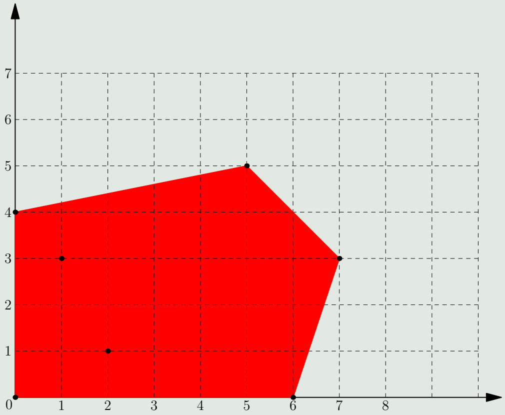
$F(x,y) = 4x^7y^3 - 5x^5y^5 + 13x^6 - 5y^4 + 21x^2y + 13xy^3 - 71$



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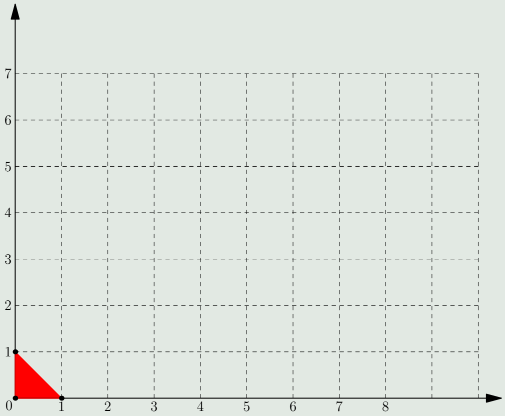
Interesting case - affine function $F(x,y) = 3x - 15y + 71$



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Theorem (BKK)

Let $F_1, \dots, F_n$ be polynomials with fixed Newton Polytopes $P_1, \dots, P_n \subset \mathbb{R}^n$ and generic coefficients. Then

$$\#\{x \in (\mathbb{C} \setminus 0)^n \mid F_1(x) = \cdots = F_n(x) = 0\} = n! V(P_1, \dots, P_n).$$

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Note that we can compute the $\deg F_i$ via the number of intersections of $X_i = \{x \in (\mathbb{C} \setminus 0)^n \mid F_i(x) = 0\}$, with a generic line.

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$NP(F) =$ convex hull of exponent vectors of a polynomial F.

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where $\ell_i(x)$ is a generic affine function. But the Newton Polytope of $\ell_i(x)$ is the standard simplex $\Delta = \text{conv}\{0, e_1, \dots, e_n\}$. And *BKK* theorem gives us

$$\deg(F_i) = n! V(P_i, \Delta[n-1]).$$

GLUE IT ALL TOGETHER!

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I. Soprunov & A.Z.; 2016

Fix integer $2 \le r \le n$ and let Δ any n-dimensional simplex, then

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- $V(K_1,\dots,K_r,\Delta[n-r]) \le V_n(\Delta)$ by monotonicity.

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Fix an integer $2 \leq r \leq n$ and let $D \subset \mathbb{R}^n$ be a convex body which satisfies

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- There are indecomposable bodies for which the inequality is not true: $D = B_1^3$.

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  Idea of a proof: An approach is similar which was used in approach to Mahler conjecture and points with positive curvature by, A. Stancu / S. Reisner, C. Schuett and E. Werner: play with a little cap around such a point.

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- In particular, this gives a complete solution in $\mathbb{R}^3$.

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C. Saroglou, I. Soprunov & A.Z.; 2017+

Let D be an n-dimensional convex body which satisfies

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for all convex bodies $K_1,\dots,K_{n-1} \subset \mathbb{R}^n$. Then D is an n-simplex!

- The above gives an affirmative answer to Question 1 in the case $r = n-1$ (we still need to push it to $r = 2$).
- In particular, this gives a complete solution in $\mathbb{R}^3$.

The idea of the proof is based on new way to perturb a convex body and a very careful study of the boundary structure of a body D.

Question 1 ($r = 2$):

Let $D \subset \mathbb{R}^n$ be a convex body which satisfies

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Substituting the above calculations in inequality in Question 1, we get

$$\frac{n}{n-1} V_{n-2}(D|(\xi, \nu)^\perp)V_n(D) \leq V_{n-1}(D|\xi^\perp)V_{n-1}(D|\nu^\perp).$$

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Giannopoulos, Hartzoulaki & Paouris; 2002.

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Moving towards Question 2 & connections to projections

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Assume Z_1, Z_2 are zonoids, then

$$V(Z_1, Z_2, D[n-2])V_n(D) \leq \mathbf{2} V(Z_1, D[n-1]) \cdot V(Z_2, D[n-1])$$

for any convex, symmetric body D.

I. Soprunov & A.Z.; 2016

Suppose D is a convex body in $\mathbb{R}^n$ and $Z_1, \dots Z_r$ are zonoids then

$$V(Z_1, \dots, Z_r, D^{n-r}) V_n(D)^{r-1} \leq \frac{r^r}{r!} \prod_{i=1}^{r} V(Z_i, D^{n-1}),$$

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Idea of the proof: Use ideas of Giannopoulos, Hartzoulaki; 2002 & Paouris / Fradelizi, Giannopoulos & Meyer; 2003: apply the Berwald's Lemma to prove that if $D \subset \mathbb{R}^n$ is a convex body, then

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Next use multi-linearity and other properties of mixed volume to bring it back to zonoids.

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There exists a constant $c_{n,r} \leq n^r r^r / r!$ such that

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holds for all convex bodies $K_1, \dots, K_r$ and D in $\mathbb{R}^n$. Moreover $c_{n,r} \leq n^{r/2} r^r / r!$ when $K_1, \dots, K_r$ are symmetric with respect to the origin.

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I. Soprunov, A.Z.; 2016 / S. Artstein-Avidan, D. Florentin & Y. Ostrover; 2014

Assume K_1, K_2, D are convex bodies in $\mathbb{R}^2$ (i.e. **Not necessary symmetric!**) then

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