§1 "Schur's Lemma"

Finite set \( R \cap G \), finite group.

Write \( r \sim s \) if \( r \) and \( s \) are in the same \( G \)-orbit and \( (r,s) \sim (t,u) \) if \( (r,s) \) and \( (t,u) \) are in the same \( G \)-orbit.

For a field \( F \), if \( F R \) the permutation module.

\[ \text{Schur's Lemma. } \quad \text{End}_{F[G]}(F R) \text{ has basis } \{ \xi_{r,s} \mid (r,s) \in R \mod G \} \]

where

\[ \xi_{r,s}(t) = \sum_{(u,t) \sim (r,s)} u \quad \text{for } t \in F \]

Moreover,

- \( \xi_{r,s} \xi_{t,u} = \sum_{(v,w) \in R \mod G} \# \{ x \in R \mid (v,x) \sim (r,s), (x,w) \sim (t,u) \} \xi_{v,w} \)
- orthogonal idempotent decomposition \( 1 = \sum_{r \in R \mod G} \xi_{r,r} \)
- anti-automorphism \( \sim : \xi_{r,s} \mapsto \xi_{s,r} \)
§2. Schur algebra

Fix $n \in \mathbb{N}_0$, $d \in \mathbb{Z}_0$.

$R = R(n,d) = \{ r = r_1 \ldots r_d \mid r_1, \ldots, r_d \in \{ 1, \ldots, n \} \}$

$\Lambda = \Lambda(n,d) = \{ (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}_0^n \mid \lambda_1 + \ldots + \lambda_n = d \}$

$\Lambda^+ = \Lambda^+(n,d) = \{ \lambda \in \Lambda(n,d) \mid \lambda_1 \geq \ldots \geq \lambda_n \}$

Dominance order on $\Lambda^+$: $\lambda \triangleright \mu \iff \sum_{r=1}^s \lambda_r \geq \sum_{r=1}^s \mu_r$

for all $s = 1, \ldots, n$.

$\alpha \in \Lambda^+ \mapsto r^\alpha = r_1^{\lambda_1} \ldots r_n^{\lambda_n} \in R$

$R \otimes \Sigma_d$ via $r^\alpha \cdot r^\beta = r_{\sigma(1)} \ldots r_{\sigma(d)}$ with orbit reps $\{ r^\alpha \mid \alpha \in \Lambda \}$.

Def. $S = S(n,d) = S^F(n,d) := \text{End}_{F\Sigma_d}(F R)$.

$\mathcal{V} = F^n$ with std basis $e_1, \ldots, e_n$, then $\mathcal{V} \otimes \mathcal{V}$ has basis $\{ e_1 \otimes e_1, \ldots, e_d \otimes e_d \mid e \in \mathcal{V} \}$, the symmetric group $\Sigma_d$ acts on the right on $\mathcal{V} \otimes \mathcal{V}$ by place permutations in pure tensors and we can identify $F R$ with $\mathcal{V} \otimes \mathcal{V}$.

Schur's Lemma: $S$ has basis $\{ \xi_{r,s} \mid (r,s) \in R^2 / \Sigma_d \}$ with Schur's product rule, antilinear operation $\gamma: \xi_{r,s} \mapsto \xi_{s,r}$, and orthogonal idempotent decomposition $1 = \sum \xi_{\alpha} \alpha$, where $\xi_{\alpha} = \xi_{r_1} \ldots \xi_{r_d}$. 
Remarks: 1) Let $d = n$, $\omega = (1, \ldots, 1) \in \Lambda$

Then $\xi_\omega S \xi_\omega \cong F\Sigma_d$, and we have an exact functor $V \mapsto \xi_\omega V$, $S$-mod $\to F\Sigma_d$-mod (equivalence $\Leftrightarrow d = 0$ or $p > d$) James' maxim.

2) Let $n < \mathbb{N}$, and $\xi_n^N = \sum_{\alpha \in \mathbb{N}^n \setminus \{0\}} \xi_{\alpha} \mathbb{N}_{\alpha}$

Then $\xi_n^N S(n,d) \xi_n^N \cong S(n,d)$, and we have an exact functor $V \mapsto \xi_n^N V$, $S(n,d)$-mod $\to S(n,d)$-mod (equivalence $\Leftrightarrow d \leq n$)

§3. Dual constructor (F-infinite). Let $A(n)$ be the algebra of polynomial functions on $GL_n(F)$, i.e., $A(n) = F[cr_{,s}]_{r,s \in \mathbb{N}^n}$ where $cr_{,s}(g) = g_{r,s}$.

The degree $d$ polynomial functions $A(n,d)$ have bases $\{cr_{,s} : cr_{r,s} \cdots cr_{r,s_d} \} (r,s) \in \mathbb{N}^n \times \mathbb{N}^d$

Product on $GL_n(F) \to$ coproduct on $A(n)$ with

$$\Delta (cr_{,s}) = \sum_{\ell \in \mathbb{N}} cr_{,\ell} \otimes cr_{,s}$$

in particular, $A(n,d)$ is a subcoalgebra.

The dual algebra $A(n,d)^* \cong S(n,d)$ with $\{\xi_{r,s}\}$ being the dual basis of $\{cr_{,s}\}$.
In particular, $A(n,d)$ is naturally an $S$-module (dual regular module) via
$\varepsilon \cdot C = \sum_k \varepsilon(k') C_k \quad \text{if} \quad \Delta(C) = \sum_k C_k \otimes C_k$.

Consequence: $S(n,d)\text{-mod} = \{ \text{degree d polynomial representations at } GL_n(F) \}$

§4 Quasihereditary algebras. A fundamental fact is that $S$-mod is a highest weight category, or, equivalently, $S$ is a quasihereditary algebra (CPS). In fact, $S$ is even a quasihereditary algebra with a standard antiinvolution. It will give a slightly unusual definition of what this means (equivalent to the standard one).

Def. Let $A$ be a f.d. algebra with antiinvolution $\tau$, and $I$ be a subset. Then $A$ is $\tau$-hereditary (wrt $I$) if there are subsets $X(i) \subseteq A$ (i \in I) s.t.

(a) $\bigcup_{i \in I} \{ x \tau(x') \mid x, x' \in X(i) \} = \text{basis of } A$

Set $A^\tau_i := \text{span} \{ x \tau(x') \mid x, x' \in X(j) \text{ for } j > i \}$.

(b) $\forall i \in I, x \in X(i), a \in A,$

\[ a x = \sum_{x' \in X(i)} E_{x, x'}(a) x' \quad (\text{mod } A^\tau_i) \]
(c) \forall i \in I \exists e_i \in \mathcal{X}(i) \text{ s.t. } \zeta(e_i) = e_i \text{ and } \forall x \in \mathcal{X}(i), j \in I:
\begin{align*}
e_i x &= \delta_{x,e_i} x \\
x e_i &= x \\
e_j x &= x \text{ or } 0
\end{align*}

Remarks: 0). \( A^{(i)} \) is an ideal and \( \Delta(i) = \langle x + A^{(i)} | x \in \mathcal{X}(i) \rangle \) is a left module with basis \( \{ v_x | x \in \mathcal{X}(i) \} \) and the action \( \alpha \cdot v_x = \sum_{x' \in \mathcal{X}(i)} f_{x'}(\alpha) v_{x'} \). Standard modules have the following properties:

1). \( \text{End}_A(\Delta(i)) = F \)

2). \( \text{Hom}_A(\Delta(i), \Delta(j)) \neq 0 \Rightarrow i \leq j \)

3). Each \( \Delta(i) \) has simple head, \( L(i) \), and \( \{ L(i) \} \) is a complete irredundant set of irreducible \( A \)-modules.

4). \( \forall i \in I \) a projective \( A \)-module \( P \) and a surjection \( P \rightarrow \Delta(i) \) whose kernel has filtration with subfactors of the form \( \Delta(j) \), \( j > i \).

Properties 1)-4) mean that \( A\text{-mod} \) is a h/w category.


$S$ is quasihereditary

A $\lambda$-tableau is a function $[\lambda] \rightarrow \{1, \ldots, n\}$ (thought of as filling boxes with numbers). A tableau is row std if its entries increase weakly along the rows; column std if row std + col std.

Given any tableau $T$, define $\pi^T \in S_n$: the word obtained by reading the entries of $T$ along the rows as reading a book.

Theorem (Green'93). $S$ is a quasihereditary algebra with antilinear involution $\tau$ with:

$X(\lambda) = \{ x^T, r^\lambda \mid T \in \text{Std}(\lambda) \}$

$(\lambda \in \Lambda^+)$

The main part of Green's theorem is that

$\bigcup_{\lambda \in \Lambda^+} \{ x^T, r^\lambda x^s, y^s \mid s, T \in \text{Std}(\lambda) \}$

is a basis of $S$. Green's student Woodcock obtained a straightening algorithm, which implies that standard codeterminants form a spanning set. Then it follows by counting using RSK that they form a basis.
Green's Theorem gives the standard modules $\Delta(A)$ with std. basis $\{v_T | T \in \text{Std}(A)\}$ whose formal characters are Schur's symmetric functions ($\Delta(A)$'s are all irreducible if $p=0$ or $p>d$). The action on the basis is explicit but involves Woodcock's straightening.

The dual picture has been worked out by Rota and collaborators in the 70's and is in some sense a little more natural, because the dual module

$$\nabla(A) := \Delta(A)^\vee \cong \{ c \in A(n,d) | c \cdot g = \lambda(g) c \text{, } \forall g \in B^- \}$$

where $\lambda: B^- \to F^\times$ is the inflation of the character $\lambda: T \to F^\times$ and $(c \cdot g)(g') = c(g'g)$.

Rota et al construct an explicit basis $\{w_T | T \in \text{Std}(A)\}$ of $\nabla(A)$ as explicit products of determinants $(\det(i) \det(j))$ determinants.

The bases $\{v_T\}$ and $\{w_T\}$ are not dual to each other (Desarmamien mix.)
Explain on example:

\[ 1 \ 2 \ 3 \]
\[ 2 \ 3 \ \Box \]

**Basis** \( \{ m_T | T \in \text{ColStd}(\lambda) \} \)

\[ \Lambda^2 V \rightarrow V \otimes d \rightarrow S^3 V \]

\[ \Lambda^2 V \otimes \Lambda^2 V \otimes V \rightarrow V^\otimes 5 \]

**Basis** \( \{ n_T | T \in \text{RowStd} \} \)

\[ n_{112} = e_1^2 e_2 \otimes e_1 e_3 \]

**Theorem**

(i) \( \text{Hom}_S (\Lambda^2 V, S^3 V) = \text{F}. \ y_\lambda \)

(ii) \( \text{Im} \ y = \nabla(\lambda) \)

(iii) \( y(m_T) = W_T \) if \( T \in \text{Std}(\lambda) \)

(iv) If \( \{ k_T | T \in \text{ColStd}(\lambda) \} \) is Lusztig's dual canonical basis at \( \Lambda^2 V \) and \( \{ e_T | T \in \text{Std}(\lambda) \} \) is Lusztig's dual canonical basis at \( \nabla(\lambda) \), then

\[ y(k_T) = \begin{cases} e_T & \text{if } T \in \text{Std}(\lambda) \\ 0 & \text{otherwise} \end{cases} \]
§8.4 Categorification

\[ V = \bigoplus_{r \in \mathbb{Z}} C e_r \rightarrow V = \bigoplus_{r \in \mathbb{Z}} C e_r \]

\[ \Lambda_{\alpha'} V \rightarrow \nabla(\lambda) \rightarrow S^3 V \]

\[ m_T \]
\[ k_T \]

\textit{translation modules, parabolic category,} $M(T)$, \textit{Verma modules,} $K(T)$, \textit{irreducible modules,} $L(T)$, \textit{finite dimensional reps over a finite W-algebra, corresponding to a nilpotent orbit Jordan shape \( \alpha \).}

\[ \mathcal{O}^\lambda(\text{gld}) \]

\[ \mathcal{O}^\lambda(\text{gld}) \]

\[ \mathcal{O}(W(\alpha, \lambda)) \]

\textit{Exact functor,} $\mathcal{O}^\lambda(\text{gld}) \overset{\phi}{\longrightarrow} W(\text{gld}, \lambda) \text{-mod}$

with

\[ \phi(K(T)) = \begin{cases} \bigoplus L(T) & \text{if } T \text{ is std.} \\ 0 & \text{otherwise} \end{cases} \]

Passing to Grothendieck groups recovers the linear algebra picture.

[Bourdon - K. 08] Reps of shifted Yangians & finite W-algebras

(Dec 13)
Schurification

\[ g \cdot h \cdot A \xrightarrow{\text{Schurification}} S^A(n,d), \text{g.h!} \]

\[ \text{cellular } \overline{A} \xrightarrow{\text{idempotent cut}} S^\overline{A}(n,d), \text{ cellular} \]

- \( S^F(n,d) \) is the usual Schur algebra, but can iterate!

Motivation: If \( \overline{A} = Z \) a zigzag algebra and \( n \mid d \), then \( S^Z(n,d) \) describes an arbitrary block at symmetric groups up to derived equivalence. Broue's Conjecture provides such a conjectural description for blocks at finite groups, provided the defect group of a block is abelian. For non-abelian defect groups, there is no conjecture and in this sense \( S^{Z}(n,d) \) provides a first glimpse into an arbitrary defect world.

A based quasihereditary wrt \( I, X \). Importantly in key examples, \( A \) could be a hyperalgebra in which case we insist that the data \( X \) respects this structure, in particular, \( X = X_- \cup X_+ \).
$S^{A}(n,d) := (M_n(A)^{\otimes d})^{\Sigma d}$.

This algebra has an obvious analogue at Schur's bars, but it doesn't have to be quasihereditary.

Suppose $A$ is defined over integers (based approach handy), define an appropriate notion of $X$-standard tableau and use it to define an analogue of codeterminants:

$$\{ Y^{\frac{A}{S_I}}_{S,T} | 2 \in \Lambda^X(n,d) | S, T \in \text{Std}^X(n,d) \}.$$  

Theorem (K-Muth'17) Let $d \leq n$. The

$$S^{A}(n,d) := \text{span}_{\mathbb{Z}}\{ Y^{\frac{A}{S_I}}_{S,T} \} \subseteq S^{A}(n,d)_{\mathbb{Z}}$$

is a full sublattice, closed under multiplication, and unital. Moreover, it is a based quasihereditary $\mathbb{Z}$-algebra.

Extending scalars to $F$ gives a based $g.l.$ $F$-algebra.

New combinatorics, interesting decomposition numbers.