Character ratios for finite groups of Lie type

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A **Character ratio** for G a finite group is $\frac{\chi(g)}{\chi(1)}$ for $\chi \in \mathrm{Irr}(G)$ or $\mathrm{IBr}(G)$.

Applications of character ratios come via: If $C_1, \ldots, C_d$ are conjugacy classes in G, the number of solutions $(x_1, \ldots, x_d)$ to $x_1 \cdots x_d = z$ for $x_i \in C_i$ is

$$\frac{\prod |C_i|}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(c_1) \cdots \chi(c_d)\overline{\chi(x)}}{\chi(1)^{d-1}}, \tag{1}$$

where $c_i \in C_i$, a classical result going as far back as Frobenius.

1 Applications

1) Counting points in representation varieties

$$\mathrm{Hom}(\Gamma, G),$$

for Γ finitely presented.

Example

$$\Gamma = T_{abc} = \langle x, y, z \mid x^a = y^b = z^c = xyz = 1 \rangle.$$

Count solutions to equation (1) with $z = 1$ over classes of order a, b, and c.

2) Random walks:

$$G = \langle C \rangle, \quad C = x^G.$$

We look at a random walk

$$1 \to c_1 \to c_1 c_2 \to \cdots$$

This is a Markov chain with eigenvalues given by character ratios $\frac{\chi(x)}{\chi(1)}$ for $\chi \in \mathrm{Irr}(G)$.

$$P_k(g) = \text{ probability at } g \text{ after } k \text{ steps.}$$

Usually $P_k \to U$. How fast?

Diaconis-Shahshahani:

$$\|P_k - U\|^2 = \left(\sum_{g \in G} |P_k(g) - U(g)| \right)^2 \leq \sum_{\chi \neq 1} \left| \frac{\chi(x)}{\chi(1)} \right|^{2k} \chi(1)^2.$$

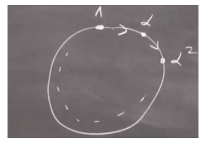
3) McKay graphs:

For G a finite group, α a character, we define a graph

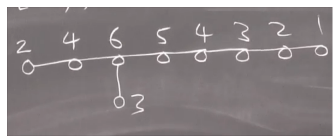
$$\Gamma(G,\alpha)$$

with vertices given by $\mathrm{Irr}(G)$, and directed edges $\chi \to$ constituents of $\chi \otimes \alpha$.

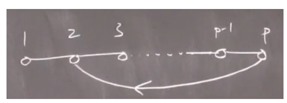
Example 1) $G = C_n$, α linear character generator:



2) $G = SL_2(5)$, α having degree 2:



3) $G = SL_2(p)$, $\alpha =$ 2-dimensional $\mathbb{F}_p^2$ natural module:



These are called McKay graphs due to the **McKay correspondence**: For G a finite subgroup of $SU_2(\mathbb{C})$, and α a 2-dimensional representation, we have

$$\Gamma(G,\alpha) = \widetilde{A}, \widetilde{D}, \widetilde{E}.$$

Theorem 1.1 *(Burnside-Brauer) If α is faithful, then every $\chi \in \mathrm{Irr}(G)$ appears in $\alpha^{\otimes n}$ for some*

$$n \leq \underbrace{\#\{\alpha(g) : g \in G\}}_{N}.$$

Define $\mathrm{diam}(G,\alpha) = \mathrm{diam}(\Gamma(G,\alpha)) \leq 2N$. Clearly

$$\mathrm{diam}(G,\alpha) \geq \frac{\log(\text{maximal degree})}{\log \alpha(1)}.$$

Example For $G = S_n$, $\alpha = \chi^{(n-1,1)}$: we have $n \geq \mathrm{diam} \geq \frac{n}{2}$.

For $G = G(q)$, $\alpha = St$ Steinberg character: $\mathrm{diam}(G, St) = 2$ with one exception when $G = U_n(q)$ (Heide-Saxl-Tiep-Zalesski).

2 Results

Theorem 2.1 *(Gluck) For $G = G(q)$, $\chi \in \mathrm{Irr}(G)$,*

$$\frac{|\chi(g)|}{\chi(1)} < \frac{3}{\sqrt{q}}.$$

The setting for the next result by Bezrukavnikov-Liebeck-Shalev-Tiep (2016) is: If $G = G(q) = \overline{G}^F$ for $\overline{G}$ a simple algebraic group, and a Levi L of $\overline{G}$, define

$$\alpha(L) = \max\left(\frac{\dim u^L}{\dim u^{\overline{G}}} : u \neq 1 \text{ unipotent}\right)$$

where u^L denotes the conjugacy class of L, etc.

Example If $\overline{G} = SL_3$ and

$$L = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} = GL_2,$$

then $\alpha(L) = \frac{2}{4} = \frac{1}{2}$.

We have $\alpha(T) = 0$ for T a torus.

Say L is split Levi if $L^F \leq P^F$, with P parabolic.

Theorem 2.2 *(Bezrukavnikov-Liebeck-Shalev-Tiep 2016)*

Suppose $G = G(q)$ (p a good prime) is simply connected. Let $x \in G$ and suppose $C_G(x) \leq L^F$, split Levi. Then for all $\chi \in \mathrm{Irr}(G)$

$$\chi(x) < \chi(1)^{\alpha(L)} \cdot f(r)$$

where $r = \mathrm{rk}(\overline{G})$.

For $G = SL_n$, $f \sim n!$.

Example

1) $G = SL_3(q)$, the theorem applies to all x except unipotent elements and regular semisimple elements with centralizer order $q^2 + q + 1$.

   For the remaining elements, we have

$$\left|\frac{\chi(x)}{\chi(1)}\right| < \chi(1)^{-\frac{1}{2}} \cdot c.$$

2) For $G = GL_n(q)$:

$$L = \prod_{i=1}^{t} GL_{n_i}(q) \qquad n_1 \geq n_2 \geq \cdots$$

   we have

$$\frac{n_1 - 1}{n - 1} \leq \alpha(L) \leq \frac{n_1}{n}$$

3) $G = E_8(q)$

L	E_7	D_7	$\cdots$	most
$\alpha(L)$	$\frac{17}{29}$	$\frac{9}{23}$		$\leq \frac{1}{4}$

3 Random Walk on E8(q)

For $G = E_8(q)$, for $x \in G$, $C_G(x)$ contained in a split Levi

$$\|P_k - U\|^2 \leq \sum_{x \neq 1} \left|\frac{\chi(x)}{\chi(1)}\right|^{2k} \chi(1)^2 \leq \sum \chi(1)^{2k(-1+\alpha)+2}$$

For $\alpha = \frac{17}{29}$, $k = 3$, this equals

$$\sum_{\chi \neq 1} \chi(1)^{-2/29} \to 0.$$

Liebeck-Shalev:

$$\sum_{\chi \in G(q)} \chi(1)^{-s} \to 1, \quad s > \frac{2}{h}.$$

For E_8, h is equal to 30, hence

$$\mathrm{Mix}(E_8(q), x^G) \leq 3.$$

4 Remaining results

Liebeck-Shalev-Tiep: $G = SL_n(q)$, $x \in G$.

Define $s =$ codimension of largest eigenspace of x over $\overline{\mathbb{F}}_p$.

Example Say x is unipotent, a sum of t Jordan blocks,

$$x = \sum_{i=1}^{t} J_{n_i}, \quad s = n - t.$$

Theorem 4.1 *For all* $\chi \in \mathrm{Irr}(G)$,

$$\frac{|\chi(x)|}{\chi(1)} < \frac{1}{q^{\gamma s}} f(n),$$

with $\gamma \approx \frac{1}{9}$.

Recall G simple, $\alpha \in \mathrm{Irr}(G)$,

$$\mathrm{diam}(G, \alpha) = \min(k : \mathrm{Irr}(G) \subset \alpha \cup \cdots \cup \alpha^k).$$

We define

$$D(G) = \max_{\alpha} \mathrm{diam}(G, \alpha).$$

Theorem 4.2 *(Liebeck-Shalev)*

For C a conjugacy class of G, $\mathrm{diam}(G, C) \leq \beta \dfrac{\log |G|}{\log |C|}$.

Conjecture 4.3

$$\mathrm{diam}(G, \alpha) \leq \delta \frac{\log |G|}{\log \alpha(1)}.$$

Theorem 4.4 *For $G = SL_n(q)$, $D(G) \leq cn$ provided $q > f(n)$ (here $c \sim 50$).*

Proof Know $\mathrm{Irr}(G) \subset St^2$.

So we aim to show $St \subseteq \chi^{cn}$ for all $\chi \in \mathrm{Irr}(G)$.

$$\langle \chi^\ell, St \rangle = \frac{1}{|G|} \sum_{g \in G_{ss}} \pm \chi^\ell(g) |C_G(g)|_p = \frac{\chi^\ell(1)}{|G|} \sum \left(|G|_p + \sum_{g \neq 1} \left(\frac{\chi^\ell(g)}{\chi^\ell(1)} \right) |C_G(g)|_p \right).$$

Now use the bound for the character ratios $\dfrac{\chi(g)}{\chi(1)}$.