CoTHH for coalgebras

\[ \text{Motivation:} \ (\text{to be made precise later.}) \]
Loop spaces.

The free loop: \( L_X := \text{maps}(S^1, X) \)

The standard loop: \( \Omega X := \text{maps}_*(S^1, X) \)

Theorem (Bökstedt–Waldhausen '87) For \( X \) 1-connected,

\[ \text{THH}(\Sigma^\infty L X) = \Sigma^\infty L X. \]

This talk: a \ Covenant/\ version for coTHH.

Theorem (Kuhn '04, Malkiewich '17, HS '18)
FCW 1-connected 1-connected \( \Rightarrow \) weaker condition to be stated.

\[ \Sigma^\infty L X \simeq \text{coTHH}(\Sigma^\infty X) \]

Why is this an improvement?

- \( \Sigma^\infty X \) has simpler models than \( \Sigma^\infty L X \)
- weaker hypotheses on \( X \)

Connections to algebraic K-theory:
\[ A(X) := \text{k}(\Sigma^\infty L X) \xrightarrow{\text{free}} \text{THH}(\Sigma^\infty L X) \]

Waldhausen K-theory \( \Rightarrow \) algebraic K-theory (hard to compute!)

\[ \text{K}(\Sigma^\infty L X) \simeq \text{K}(\Sigma^\infty X) \quad \xrightarrow{\text{if}} \quad \text{coTHH}(\Sigma^\infty X) \]

HS '16
§ Coalgebras. Primary interest: differential graded setting or spectra.

Coalgebra: cocomonoid in symmetric monoidal category equipped with counital, cocommutative multiplication.

E.g. spaces, (Top, x, *). If \( * \maps X \xrightarrow{\Delta} X \times X \) is a coalgebra structure:

\[
\begin{array}{ccc}
X \times X & \xrightarrow{\text{Id} \times \text{Id}} & X \times X \\
\downarrow \Delta & & \downarrow \text{exId} \\
X & \xrightarrow{\Delta} & X \times X \\
\end{array}
\]

forces the comultiplication to be the diagonal.

So any space is a coalgebra via the diagonal, and thus is the unique coalgebra structure. Note in this case it's cocommutative.

- Chain complexes. \((CH_* \otimes k)\) for \(k\) a field.

1. \( C = C_*(X, k) \). Comultiplication on \( X \maps \text{cornut. on chains} \)

\[
C_*(\ast) \leftarrow C_*(X, k) \xrightarrow{\Delta} C_*(X) \otimes C_*(X)
\]

\[
C_*(X \times X)
\]

2. \( R \) a finite type dga (differential graded algebra)

\[
R \xrightarrow{} (R \otimes R) \xrightarrow{} R \otimes R
\]

(Not necessarily comm. dga, nor necessarily cocomm. coalg.)
Examples continued.

3. \((\text{Spectra}, \ A, \ S) \times \text{a space as in } \Theta. \)

\[
\Sigma_+^\infty(X_+) \quad \Sigma_+^\infty(e) \quad \Sigma_+^\infty(X \times X) \cong \left( \Sigma_+^\infty X \right) \wedge \left( \Sigma_+^\infty X \right)
\]

(i) \(S = \sum_+^\infty \leftarrow \Sigma_+^\infty X \rightarrow \Sigma_+^\infty (X \times X) \cong \left( \Sigma_+^\infty X \right) \wedge \left( \Sigma_+^\infty X \right)\)

Based on diagonal, so automatically coassociative.

(ii) [Example works more generally.]

\(f: A \to B \text{ comm. rings.} \)

\(B \wedge_A B \text{ is a } B\text{-coalgebra as follows:} \)

\[
B \wedge_A B = B \wedge_A A \wedge_A B \xrightarrow{\text{inf}} B \wedge_A^B B \wedge_A B \xrightarrow{\delta} (B \wedge_A B) \wedge_B (B \wedge_A B)
\]

(iii) [Instance of (ii)]

\(S \to \text{HFF}_p \quad \text{HFF}_p \wedge_S \text{HFF}_p \quad \text{dual Steenrod algebra.} \)

(iv) \(R \text{ a compact ring spectrum} \)

\(DR = \text{map}(R, S) \) is the Spanier-Whitehead dual.

As for dga's:

\(DR \to D(R \wedge R) \leftarrow DR \wedge DR\)

co-algebra up to homotopy.
(Strict) Symmetric monoidal categories of spectra:

in those that are known, rings and modules work well.
Issues with commutativity up to homotopy (for e.g.) don’t arise.

Examples of such cats:

\[
\begin{align*}
&\{\text{Symmetric spectra} \} \\
&\{\text{Sp} \} \\
&\{\text{Sp}^\Sigma \} \\
&\{\text{general} \} \\
&\{\text{EKMM} \} \\
&\{\text{S-mods} \}
\end{align*}
\]

\(\mathcal{S}p \in \mathcal{E}\)

we’ll use \(\mathcal{S}p\) for one of the cats.

Homotopically, coalgebras spectra are not understood well in \(\mathcal{S}p\).

Prop (Pérez-S, 19) Any \(\mathcal{S}\)-coalgebra in \(\mathcal{S}p\) is co-commutative.

\(\text{Pf.}\) \(\mathcal{S} \land C \rightarrow C\)

\(\text{Surj}\) therefore this is co-commutative.

So coalgebras are not modeled well here. Should move to \(\mathcal{S}\)-cats or find a new model.

Note: in ongoing work of Bowman-der Waard, developing this. Not for today’s talk.

& Def of \(\text{coTHH}\).

\(\text{Defn.}\) For \((\mathcal{C}, \Sigma, \Delta)\) a coalgebra, the \(\text{co-Hochschild complex}\)

(or cyclic co-bar construction) is a cosimplicial object with

\[
(\text{coTHH}(\mathcal{C}))^n = 
\begin{cases} 
\mathcal{C} & n = 0 \\
\mathcal{C} \otimes \mathcal{C}^n & n \geq 1 
\end{cases}
\]

\(d^i = \Delta \text{ in the } i\text{th spot, } i < n\)

\(s^i = \Sigma \text{ in the } i\text{th spot}\)

\(d^{n+1} = \mathcal{E} \cdot \Delta\)
\[
C \xrightarrow{\delta} C \otimes C \xrightarrow{\delta \otimes 1} C \otimes C \otimes C
\]

coTHH(C) := \holim (coTHH \cdot C)

Duality of def of THH. But note: geometric realization (= ho cd) has good properties of commutation with \( \otimes \). This is an added difficulty for coTHH.

Also, spectral sequences involved don't behave as well.

\[\text{For } (C, e, \Delta, \pi) \quad \text{II} \overset{m}{\hookrightarrow} C \quad \Delta \pi = 2 \otimes \pi\]

The cobar complex is \((S^2 \cdot C)^{\pi} = 1 \otimes C^{\pi} \otimes 1\)

\[d^i = n \text{ in } 0^{th} \quad s^n = e \quad \text{in } i^{th} \text{ spot}\]

The cobar construction \(\Omega C\) is the homotopy inverse limit \(\holim S^2 \cdot C\)

§3 DG- \((Ch_{k}, \otimes, k)\) for \(k\)-field.

Doi '81; Faram - Solotar '00; Hess - P - Scott '09

\[\text{coHH}_{\otimes}(C) = \text{Tot} \otimes \left(\text{coTHH} \cdot C\right)\]

One can show that for \(C\) a 1-connected DG,

1. \(\text{coHH}_{\otimes}(C) \simeq \text{coTHH}_{\text{Ch}_k}(C)\)

2. For \(X\) 1-connected, \(\Omega(C \mathcal{X}) \simeq C \mathcal{X}(S^2 \mathcal{X})\)

Want to take coTHH of ring spectrum, or cat of modules, and say these agree...

Properties of coTHH

\[\text{Thm (Hess 16, HPS 09)} \text{ for } C \text{ conn DG,}\]

\[\text{coHH}(C) \simeq \text{HH}(S^2 \cdot C)\]
This is at the level of homology — can we categorify?

**Thm (HS'18)**

For $C$ a connected DG coalgebra, there is a Quillen equivalence

$$ \text{Comod } C \longrightarrow \text{ Mod } L_C $$

homotopy theory developed in work of Hess, K

- Riehl - Shulman '17

+ follow-up by Garner, K

- Riehl '18

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**C**: something weakly equivalent to unit

$something \text{ w.e. to unit} \leftarrow L_C$

Using this result:

**Prop (HS'18)**: [Agreement for $\text{aHH}$]

$$ \text{coHH}_*(C) \approx \text{HH}_* (\text{dg cofree } C) $$

**Pf sketch**

$$ \text{coHH}(C) \approx \text{HH}(L_C) \approx \text{HH} (\text{dg free } L_C) $$

$$ \approx \text{HH} (\text{dg cofree } C) $$

by Keller agreement for $\text{HH}$

Note: $\text{HH}$ appears because $\text{dg cofree } C$ is a category.

Another property of $\text{HH}$: Manta invariance. Want analogue for $\text{coHH}$.

**Prop (HS'18)** If $C$ and $D$ are Manta equivalent via a braiding

(see Berglund - Hess '18), then $\text{coHH}(C) \approx \text{coHH}(D)$,

[for def, need notion of dualizability.]
Spectra.

For spectra, restrict to suspension spectra to be able to work in strict, rather than \(\infty\)-categorical, setting.

Recall: Eilenberg-Moore SS, \(\mathcal{S}X \to PX \to X\).

The SS converges strongly if \(X\) is connected and \(\pi_1 X\) acts nilpotently on \(H_*(\mathcal{S}X; \mathbb{Z})\). We will call such \(X\) EMSS-good.

E.g.: For \(X\) 1-connected, \(X\) is EMSS-good.

This is the "weaker condition" stated in theorem stated at the beginning.

Thm. (HS ’18, Kuhn ’04, Malkiewich ’17)

1. If \(X\) is EMSS-good, then \(\text{coTHH}(\Sigma^\infty_+ X) = \Sigma^\infty_+ \mathcal{S} X\).

2. If \(X\) is 1-connected, \(\Sigma_-(\Sigma^\infty_+ X) = \Sigma^\infty_+ \mathcal{S} X\).

Cor. For 1-connected \(X\), \(\text{coTHH}(\Sigma^\infty_+ X) = \text{THH}(\Sigma^\infty_+ \mathcal{S} X)\)

Categorification:

Thm (HS ’16). For \(X\) connected, there is a Quillen equivalence

\[
\begin{align*}
\text{Comod}_{\Sigma^\infty_+ X} & \cong \text{Mod}_{\Sigma^\infty_+ \mathcal{S} X} \\
\Sigma^\infty_+ X & \cong \Sigma^\infty_+ \mathcal{S} X
\end{align*}
\]
From this, get agreement:

Cor (HS '18) For X 1-connected,

\[ \text{co} \text{THH}(\Sigma^\infty_+ X) \cong \text{THH}(\text{Thick}_{\Sigma^\infty_+ X}(s)) \]

\[ \cong \]

\[ \text{THH}(\Sigma^\infty_+ \Omega^\infty X) \cong \text{THH}(\text{Thick}_{\Sigma^\infty_+ \Omega^\infty X}(s)) \]

B-M '12

Blumberg - Mandell