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Name: Michael McNulty Email/Phone: mcnulty@math.ucr.edu/6099821991

Speaker's Name: Xuwen Zhu

Talk Title: Ps.d.05 lecture 1

Date: 9/3/19 Time: 9:30 am/ pm (circle one)

Please summarize the lecture in 5 or fewer sentences: This lecture introduced the notion of a pseudodifferential operator. It also described classical Kohn-Nirenberg symbols, quantization, and adjoints.

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MSRI LECTURES ON PSEUDODIFFERENTIAL OPERATORS

XUWEN ZHU

ABSTRACT. Rough notes for lectures at the MSRI introductory workshop in Fall 2019.

A large part of these notes is a shortened version of lecture notes by Richard Melrose, available at http://math.mit.edu/~rbm/im190c2.ps.

1. PROLOGUE: WHY STUDY PSEUDODIFFERENTIAL OPERATORS?

In these two lectures we will mostly study pseudodifferential operators on $\mathbb{R}^n$. They do also work beautifully well on manifolds.

A very basic example of a pseudodifferential operator is the $L^2$ inverse to the shifted Laplacian

$$P = \Delta + 1, \quad \Delta := -\sum_{j=1}^{n} \partial_{x_j}^2.$$ 

This inverse is a Fourier multiplier (here $\mathcal{S}$ denotes Schwartz functions):

$$P^{-1} = Q : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n), \quad \hat{Q}u(\xi) = \frac{1}{1 + |\xi|^2} \hat{u}(\xi).$$

Now, imagine that we instead have a variable coefficient operator, e.g.

$$-\sum_{j,k} p_{jk}(x) \partial_{x_j} \partial_{x_k},$$

where $p_{jk}$ is a positive definite matrix depending on $x$. What would the inverse be? At the end of this lectures we construct an approximate inverse which is a pseudodifferential operator. This was the original motivation for studying pseudodifferential operators in the theory of PDE.

Pseudodifferential operators are a general class of operators which include differential operators, Fourier multipliers like $Q$ above, approximate inverses to elliptic differential operators, and a lot more.
2. Special quantization formula

To get the formula for a general pseudodifferential operator, we first look at a differential operator of order $m$

$$A = \sum_{|\alpha| \leq m} a_\alpha(x) D_x^\alpha, \quad a_\alpha \in C^\infty(\mathbb{R}^n),$$

where we henceforth adopt the notation

$$D_x^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}, \quad D_{x_j} = \frac{1}{i} \partial_{x_j}.$$ 

Take $u \in \mathcal{S}(\mathbb{R}^n)$ and write by the Fourier inversion formula.

$$u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \hat{u}(\xi) \, d\xi.$$ 

Now let's differentiate under the integral sign to obtain

$$D_x^\alpha u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i x \cdot \xi} \xi^\alpha \hat{u}(\xi) \, d\xi.$$ 

From here we get

$$Au(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i x \cdot \xi} a(x, \xi) \hat{u}(\xi) \, d\xi \quad (1)$$

where $a(x, \xi)$ is the symbol of the operator:

$$a(x, \xi) = \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha. \quad (2)$$

To obtain a pseudodifferential operator, we simply take a more general function $a(x, \xi)$ in (1), one which is not necessarily a polynomial in the $\xi$ variables. The corresponding operator $A$ is called the quantization of $a$, and we write

$$A = \text{Op}_0(a).$$

Here $\text{Op}_0$ stands for quantization for $y$-independent symbols (as opposed to general symbols defined later). Note that the Fourier multiplier $Q$ defined above can now be written as

$$Q = \text{Op}_0 \left( \frac{1}{1 + |\xi|^2} \right).$$

To get good properties of $\text{Op}_0(a)$ we need to make certain assumption on the behavior of $a$ at infinity. This brings us to symbol classes.
3. Classical Kohn–Nirenberg Symbols

We use the notation
\[ \langle \xi \rangle := \sqrt{1 + |\xi|^2}. \]
This is asymptotic to $|\xi|$ as $\xi \to \infty$ and is also smooth at $\xi = 0$.

**Definition 3.1.** Let $m \in \mathbb{R}$. We say $a(z, \xi) \in C^\infty(\mathbb{R}^p \times \mathbb{R}^n)$ lies in the symbol class $S^m(\mathbb{R}^p; \mathbb{R}^n)$, if for all multiindices $\alpha, \beta$

\[ |\partial_x^\alpha \partial_\xi^\beta a(z, \xi)| \leq C_{\alpha\beta} \langle \xi \rangle^{-|\alpha - \beta|} \]

Here we allow $p \neq n$ for future use, but in the original formula (1) we have $p = n$.

The derivative bounds above mean the following: $a = O(\langle \xi \rangle^m)$, differentiation in $z$ does not change the growth of $a$, but differentiation in $\xi$ gives decay by a power of $\xi$. As an exercise, you can check that:

- a polynomial of the form (2) lies in $S^m(\mathbb{R}^n; \mathbb{R}^n)$ if we assume that all derivatives of $a_\alpha$ lie in $L^\infty$, and
- the symbol of the operator $Q$ above, $a(x, \xi) = (\xi)^{-2}$, lies in $S^{-2}(\mathbb{R}^n; \mathbb{R}^n)$.

4. General Quantization Formula

And the Power of Integration by Parts

We now return to the formula (1). Let $a \in S^m(\mathbb{R}^n; \mathbb{R}^n)$. Recalling the definition of the Fourier transform, we rewrite (1) as

\[ Au(x) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} e^{i(x-y) \cdot \xi} a(x, \xi) u(y) \, dy \, d\xi; \quad u \in \mathcal{S}(\mathbb{R}^n). \]

(This only makes sense for $m < -n$, more on that later.)

We arrive to the general quantization formula by allowing $a$ to also depend on $y$. (This will be useful in deriving properties of quantization.) Namely, for $a \in S^m(\mathbb{R}^{2n}; \mathbb{R}^n)$ we define

\[ \text{Op}(a) u(x) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} e^{i(x-y) \cdot \xi} a(x, y, \xi) u(y) \, dy \, d\xi. \]  \hfill (3)

For $m < -n$, the integral in (3) converges and we get

\[ a \in S^m(\mathbb{R}^{2n}; \mathbb{R}^n), \ m < -n \implies \text{Op}(a) : \mathcal{S}(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n). \]  \hfill (4)

We now make sense of the oscillatory integral (3) for general $a$ (in particular, for $a$ which is a polynomial of the form (2)), by integrating by parts in $y$. Let's just do it
one time. We write
\[ \text{Op}(a)u(x) = (2\pi)^{-\nu} \int_{\mathbb{R}^{2n}} (e^{i(x-y)\cdot \xi}(\xi)^{2}(\xi)^{-2}a(x,y,\xi)u(y)) \, dyd\xi \]
\[ = (2\pi)^{-\nu} \int_{\mathbb{R}^{2n}} ((1 - \xi \cdot D_{y})e^{i(x-y)\cdot \xi}(\xi)^{-2}a(x,y,\xi)u(y)) \, dyd\xi \]
\[ = (2\pi)^{-\nu} \int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot \xi}(1 + \xi \cdot D_{y})(\xi)^{-2}a(x,y,\xi)u(y)) \, dyd\xi. \]
(5)

The integration by parts in the third line above does make sense for \( a \in S^{m} \) and \( m < -\nu \). However, in the last line we compute
\[ (1 + \xi \cdot D_{y})(\xi)^{-2}a(x,y,\xi)u(y)) = O((\xi)^{m-1}(y)^{-\nu}) \]
so the last integral actually converges when \( m < 1 - \nu \), which is better than the original definition of quantization! We can now integrate by parts repeatedly to arrive to
\[ a \in S^{m}(\mathbb{R}^{2n}, \mathbb{R}^{n}), \text{ any } m \implies \text{Op}(a) : \mathcal{S}(\mathbb{R}^{n}) \to L^{\infty}(\mathbb{R}^{n}). \] (6)

Some caution is needed here: what we really mean is that the linear operation \( \text{Op}(a) \) is extended to the symbol class \( S^{m} \) by continuity from, say, \( S^{m-1} \). Such an extension is necessarily unique, and we can prove identities for \( a \in S^{m} \) by just proving them for rapidly decaying \( a \) and arguing by approximation. (There are some subtleties here regarding "approximating by nice symbols").

We can upgrade (6) further as follows:
\[ a \in S^{m}(\mathbb{R}^{2n}, \mathbb{R}^{n}), \text{ any } m \implies \text{Op}(a) : \mathcal{S}(\mathbb{R}^{n}) \to \mathcal{S}(\mathbb{R}^{n}). \] (7)

For that we need to apply the operators \( x_{j}, D_{x_{j}} \) to \( \text{Op}(a) \) and use the identities
\[ x_{j}\text{Op}(a) = \text{Op}(a)x_{j} - \text{Op}(D_{x_{j}}a), \]
\[ D_{x_{j}}\text{Op}(a) = \text{Op}(a)D_{x_{j}} + \text{Op}(D_{x_{j}}a + D_{y_{j}}a) \]
the first of which is proved by integrating by parts in \( \xi_{j} \) and the second one, by integrating by parts in \( y_{j} \). We show the first one in a bit more detail since it will be used again later:
\[ (x_{j}\text{Op}(a) - \text{Op}(a)x_{j})u(x) = (2\pi)^{-\nu} \int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot \xi}(x_{j} - y_{j})a(x,y,\xi)u(y) \, dyd\xi \]
\[ = (2\pi)^{-\nu} \int_{\mathbb{R}^{2n}} (D_{x_{j}}e^{i(x-y)\cdot \xi})a(x,y,\xi)u(y) \, dyd\xi \]
\[ = -(2\pi)^{-\nu} \int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot \xi}D_{x_{j}}a(x,y,\xi)u(y) \, dyd\xi. \] (8)
Since \( D_{x_{j}}a, D_{x_{j}}a, D_{y_{j}}a \) still lie in \( S^{m} \), we see that \( x_{j}\text{Op}(a), D_{x_{j}}\text{Op}(a) : \mathcal{S}(\mathbb{R}^{n}) \to L^{\infty}(\mathbb{R}^{n}) \), and iteration gives \( x^\alpha D_{x}^\beta \text{Op}(a) : \mathcal{S}(\mathbb{R}^{n}) \to L^{\infty}(\mathbb{R}^{n}) \), which implies (7).

The above discussion leads to the following statement:
Proposition 4.1. Assume that $a \in S^m(\mathbb{R}^{2n}; \mathbb{R}^n)$. Then we may define

$$\text{Op}(a) : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n), \quad \text{Op}(a) : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n).$$

Here the additional powers of $x, y$ are handled similarly to the previous argument. To get the mapping property on tempered distributions, we use a definition by duality:

$$\langle \text{Op}(a)u, \varphi \rangle_{L^2} := \langle u, \text{Op}(a)^* \varphi \rangle_{L^2}, \quad u \in \mathcal{S}'(\mathbb{R}^n), \quad \varphi \in \mathcal{S}(\mathbb{R}^n)$$

where the adjoint $\text{Op}(a)^*$ has the form (3) with a different symbol (see below) and thus maps $\mathcal{S}(\mathbb{R}^n)$ to itself.

We call the resulting class of operators $\text{Op}(a)$ pseudodifferential operators. In particular, we denote by

$$\Psi^m(\mathbb{R}^n)$$

operators of the form $\text{Op}(a)$ where $a \in S^m(\mathbb{R}^{2n}; \mathbb{R}^n)$.

Define the residual operator class

$$\Psi^{-\infty}(\mathbb{R}^n) := \bigcap_{m \in \mathbb{R}} \Psi^m(\mathbb{R}^n).$$

One can show (with a bit of annoying technical work) that every element of $\Psi^{-\infty}$ has the form $\text{Op}(a)$ where $a$ lies in the residual symbol class

$$S^{-\infty}(\mathbb{R}^{2n}; \mathbb{R}^n) := \bigcap_{m} S^m(\mathbb{R}^{2n}; \mathbb{R}^n).$$

Note that $a \in S^{-\infty}$ simply means that each derivative of $a$ decays like $O((|\xi|^{-\infty})$. Then the integral kernel of (3) converges with all the $x, y$ derivatives, implying that every $A \in \Psi^{-\infty}$ is a smoothing operator:

$$Au(x) = \int K(x, y)u(y) \, dy \quad \text{where} \quad K(x, y) \in C^\infty(\mathbb{R}^{2n}).$$

In particular we have the mapping property $A : \mathcal{S}' \to C^\infty(\mathbb{R}^n)$.

5. Reduction to $y$-independent symbols

We now show that the general quantization procedure $\text{Op}$ from (3) actually gives the same class of operators as the special quantization procedure $\text{Op}_0$ from (1), and get a useful asymptotic expansion:

Theorem 1. Assume that $a \in S^m(\mathbb{R}^{2n}; \mathbb{R}^n)$. Then there exists $\bar{a} \in S^m(\mathbb{R}^n; \mathbb{R}^n)$ such that

$$\text{Op}(a) = \text{Op}_0(\bar{a}).$$

Moreover, we have the asymptotic expansion

$$\bar{a}(x, \xi) \sim \sum_{k=0}^{\infty} \bar{a}_k, \quad \bar{a}_k := \frac{1}{k!} (-i \partial_y \cdot \partial_\xi)^k a(x, y, \xi)|_{y=x}, \quad \partial_y \cdot \partial_\xi := \sum_{j=1}^{n} \partial_{y_j} \partial_{\xi_j}$$

(9)
in the following sense: for each $N$,

$$\tilde{a}(x, \xi) - \sum_{k=0}^{N-1} \tilde{a}_k \in S^{m-N}(\mathbb{R}^n; \mathbb{R}^n).$$

Note here that the expansion does make sense: the $k$-th term in the expansion is in $S^{m-k}$ due to the fact that Kohn–Nirenberg symbols improve by a power of $\xi$ when differentiated in $\xi$. (This is the first time we use this fact, actually.) Note also that the above is an asymptotic expansion, not a convergent series! Asymptotic expansions like the one above are very common in microlocal analysis.

We will not prove Theorem 1 (see Melrose’s notes for a proof). We instead prove a simpler statement (from which the full theorem follows, but after a good amount of annoying technical work):

**Proposition 5.1.** For each $N$ we may write

$$\text{Op}(a) = \text{Op}_0 \left( \sum_{k=0}^{N-1} \tilde{a}_k \right) + \text{Op}(r_N), \quad r_N \in S^{m-N}(\mathbb{R}^{2n}; \mathbb{R}^n)$$

where $\tilde{a}_k$ are the terms in the expansion (9).

**Proof.** We just show the cases $N = 1, N = 2$, with higher $N$ obtained similarly. We first do $N = 1$. The symbol

$$a(x, y, \xi) - \tilde{a}_0(x, \xi) = a(x, y, \xi) - a(x, x, \xi)$$

vanishes on the partial diagonal $\{x = y\}$. We can then write

$$a(x, y, \xi) - \tilde{a}_0(x, \xi) = \int_0^1 \partial_t (a(x, x + t(y - x), \xi)) \, dt = \sum_{j=1}^n (y_j - x_j) b_j(x, y, \xi),$$

(10)

$$b_j(x, y, \xi) = \int_0^1 (\partial_{y_j} a)(x, x + t(y - x), \xi) \, dt.$$

From the definition of $b_j$ we see that $b_j \in S^m(\mathbb{R}^{2n}; \mathbb{R}^n)$. We now use the key identity proved by (8):

$$b \in S^m(\mathbb{R}^{2n}; \mathbb{R}^n) \quad \Rightarrow \quad \text{Op}((y_j - x_j)b) = \text{Op}(D_{\xi_j}b).$$

(11)

We get then

$$\text{Op}(a) - \text{Op}_0(\tilde{a}_0) = \text{Op}(r_1), \quad r_1(x, y, \xi) := \sum_{j=1}^n D_{\xi_j} b_j(x, y, \xi)$$

and $r_1$ does lie in $S^{m-1}(\mathbb{R}^{2n}; \mathbb{R}^n)$ owing to the differentiation in $\xi$.

To do $N = 2$, we iterate this process further, applying it now to the symbol $r_1$. We see that the next term in the expansion should be the restriction of $r_1$ to $\{x = y\}$;
indeed, \( r_1(x,y,\xi) - r_1(x,x,\xi) \) can be again written in the form (10). It is easy to compute that
\[
r_1(x,x,\xi) = -i(\partial_y \cdot \partial_\xi)a(x,y,\xi)|_{y=x} = \tilde{a}_1(x,\xi).
\]
So we get
\[
\text{Op}(a) - \text{Op}_0(\tilde{a}_0 + \tilde{a}_1) = \text{Op}(r_2), \quad r_2 \in S^{m-2}(\mathbb{R}^{2n};\mathbb{R}^n).
\]
\[\square\]

For \( A = \text{Op}(a) \in \Psi^m(\mathbb{R}^n) \), where \( a \in S^m(\mathbb{R}^{2n};\mathbb{R}^n) \), we define the principal symbol \( \sigma^m(A) \) as follows:
\[
\sigma^m(A) = [a(x,x,\xi)] \in \frac{S^m(\mathbb{R}^n,\mathbb{R}^n)}{S^{m-1}(\mathbb{R}^n,\mathbb{R}^n)}.
\]  \hspace{1cm} (12)

The principal symbol will have nice algebraic properties as we will see soon. What we see immediately from Proposition 5.1 is the following statement: if \( A \in \Psi^m(\mathbb{R}^n) \), then
\[
\sigma^m(A) = 0 \iff A \in \Psi^{m-1}(\mathbb{R}^n).
\]
So the principal symbol does determine \( A \) modulo a lower order term. One often suppresses the order of the operator in the notation, writing \( \sigma \) instead of \( \sigma^m \).

6. ADJOINTS

We now discuss algebraic properties of the classes \( \Psi^m(\mathbb{R}^n) \). One algebraic property that we can do easily is closure under adjoints:

**Theorem 2.** Assume that \( A \in \Psi^m(\mathbb{R}^n) \). Then \( A^* \in \Psi^m(\mathbb{R}^n) \) and \( \sigma(A^*) = \overline{\sigma(A)} \). Here the adjoint is understood in the following sense:
\[
\langle Au, v \rangle_{L^2} = \langle u, A^*v \rangle_{L^2} \quad \text{for all} \quad u, v \in \mathcal{S}(\mathbb{R}^n).
\]

**Proof.** Let \( A = \text{Op}(a) \) where \( a \in S^m(\mathbb{R}^{2n};\mathbb{R}^n) \). Then we have the following representation of the adjoint:
\[
\text{Op}(a)^*u(x) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot \xi} \overline{a(y,x,\xi)} u(y) dy d\xi.
\]
From here the result follows immediately since \( \text{Op}(a)^* = \text{Op}(a^*) \) where \( a^*(x,y,\xi) = \overline{a(y,x,\xi)} \).
\[\square\]