MSRI

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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Michael McNulty Email/Phone: mmcnulty@math.ucr.edu

Speaker's Name: Rafe Mazzeo

Talk Title: Lecture on geometric microlocal analysis

Date: 9 / 4 / 19 Time: 11 : 00 am / pm (circle one)

Please summarize the lecture in 5 or fewer sentences: This lecture discussed the construction of parametrices for the Laplace-Beltrami operator on various Riemannian manifolds and the b-calculus.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

☑ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.

☑ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.

- Computer Presentations: Obtain a copy of their presentation
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☑ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)

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MSRI LECTURES ON GEOMETRIC MICROLOCAL ANALYSIS LECTURE 2

LECTURER: RAFE MAZZEO

ABSTRACT. Rough notes for lectures on geometric microlocal analysis at the MSRI introductory workshop in Fall 2019.

- Start with noncompact space and an elliptic operator: (M, g), Δ; $\Delta u = f$
  - Want: $G \in \mathcal{D}'(M \times M)$ which solves $\Delta G = I = G\Delta$.
  - Need to understand G
  - We'll have
$$u(z) = \int G(z, \tilde{z}) f(\tilde{z}) dv(\tilde{z})$$
    otherwise known as $u = (\pi_L)_*(G\pi_R^* f)$ where π_L and π_R are projections on the left/right from $M \times M \to M$.
  - G is singular along the diagonal of $M \times M$, i.e., when $z = \tilde{z}$.
- Parametrix method allows us to find $\tilde{G}$ which approximates G; $\Delta\tilde{G} = I - R_1$, $\tilde{G}\Delta = I - R_2$ with $R_1, R_2 \in \Psi^{-\infty}$ (smoothing operators), i.e., $R_j(z, \tilde{z}) \in C^\infty(M \times M)$.
  - This gives local regularity but not Fredholmness or mapping properties.
- <u>Better:</u> $R_j : L^2 \to \rho^k H^\infty \implies \Delta$ is Fredholm
  - The kernel decays away from the diagonal
  - $\Delta u = 0$ (u in L^2-null space) $\implies u = R_2 u \implies u \in C^\infty$ and definite rate of decay.
- <u>Test case</u>: $\Delta - s(n-1-s)$ on $\mathbb{H}^n$
  - Set $\lambda = s(n-1-s)$
  - We want to solve for the resolvent $(\Delta - \lambda)^{-1}$
  - G_s solves
$$\left(\partial_r^2 + (n-1)\frac{\cosh r}{\sinh r}\partial_r + \lambda\right)G_s = 0$$
  - Setting $\rho = e^{-r}$ yields $G_s \sim \rho^s$ or ρ^{n-1-s} as $\rho \to 0$.
  - Specialize to $n = 3$: explicit solution $G_s = \dfrac{e^{-(s-1)\rho}}{\sinh\rho}$ in the UHS model
  - Dilation invariance: $G_s(x, \tilde{x}, y - \tilde{y}) = G_s(\beta x, \beta\tilde{x}, \beta(y - \tilde{y}))$ for all $\beta > 0$
  - Blow up the submanifold $\{x = 0,\ \tilde{x} = 0,\ y = \tilde{y}\}$ in $M \times M$. and define $M_0^2 = [M \times M, S]$, that is, we are attaching a sphere at each point of the diagonal

  - G is singular along the diagonal and smooth up to the front face
  - $(\Delta - s(n-1-s))u = f \in C_0^\infty \implies u = G_s f \sim x^2 + h.o.t. = a(y)x^s + \ldots + b(y)x^{n-1-s}$
- Scattering operator: $S(s) : a(y) \mapsto b(y)$; Scattering "matrix"
  - $s = \dfrac{n-1}{2}$ is a "critical line"; to the left, a decays more and to the right b decays more
- Generalize:
  - $\Delta + V - s(n-1-s)$, $V \in C_0^\infty$ or $V = 0$ in $\mathbb{H}^n \setminus \Omega$
  - Still want G_s with $(\Delta + V - s(n-1-s))G = \delta$
  - G_s is approximated by $\tilde{\chi}_1 G_{in}\chi_1 + \tilde{\chi}_2 G_{out}\chi_2$ as before
  - We get

$$\big(\Delta + V - s(n-1-s)\big)\tilde{G}_s = \tilde{\chi}_1 L G_{in}\chi_1 + \tilde{\chi}_2 L G_{out}\chi_2 + [L, \tilde{\chi}_1]G_{in}\chi_1 + [L, \tilde{\chi}_2]G_{out}\chi_2$$

  - Want to solve $LG_{in} = I - Q_1$ with $Q \in C^\infty(M \times M)$.
- Claim: $\tilde{G}_s$ is smooth on M_0^2.
  - In fact: $L\tilde{G}_s = I - R_1$ and $\tilde{G}_s L = I - R_2$ with $R_1, R_2 \in C^\infty(M_0^2)$ where C^∞ is being identified with $\mathscr{A}_{phg}$, polyhomogeneous functions with smooth expansions.
- Consider $\mathbb{H}^n/\Gamma$, convex cocompact quotients
  - $M = \mathbb{H}^n/\Gamma \to \bar{M}$, its compactification
  - Look at $M_0^2 = [\bar{M} \times \bar{M}, \partial(\text{diag})]$
  - Consider $\Delta - s(n-1-s)$ on $\mathbb{H}^n/\Gamma$
  - G_s =resolvent$\in \Psi_0^{-2,\epsilon}$ and is meromorphic in s where the subscript 0 refers to the blow-up
  - Consider the vector fields $\nu_0 = \{x\partial_x, x\partial_{y_i}\}$
  - Take $\{U_j\}$ a cover of our manifold
  - For each U_j, choose G_j an inverse for $\Delta - s(n-1-s)$ in $U_j \subset \overline{\mathbb{H}^n}$.
  - $\tilde{G}_s = \sum \tilde{\chi}_j G_j \chi_j$ and $(\Delta - s(n-1-s))\tilde{G}_s = I - R$ for a smoothing operator R
    - $[\Delta, \tilde{\chi}_j]G_j\chi_j$ shows up in the error
    - The commutator is smooth and bounded while each G_j decays like x^s which tells us that the error is supported away from the diagonal.
    - Thus, $(\Delta - s(n-1-s))\tilde{G}_s = I$+compact smoothing operator
  - We want $G_s = \tilde{G}_s(I - R_s)^{-1}$
    - Approximate $(I-R)^{-1} \sim I + R_s + R_s^2 + \ldots$
  - $g = \rho^{-2}\tilde{g}$, $\dfrac{dx^2 + h(x,y)}{x^2}$ with $h(x,y) \sim \sum[(h_l)_{ij}(y)dy^i dy^j]x^l$
    - $\Delta_g \sim \Delta_{\mathbb{H}^n}$
    - $\sum a_{j\alpha}(x,y)(x\partial_x)^j(x\partial_y)^\alpha$ the leading term of Δ in $\mathbb{H}^n$.
- Summary: start with (M, g)

- First step:
  - Compactify and form $M_0^2 = [\bar{M} \times \bar{M}, \partial(\text{diag})]$
  - $\tilde{G} = G_0 + G_1 + G_2$ operator does not degenerate after blowup
  - Lift $x\partial_x$, $x\partial_{y_i}$ to M_0^2
  - $\Delta G_0 = I - R_0$ with R_0 smooth localized along blowup
  - $R_0 : x^s L^2 \to x^s H_0^\infty$ where subscript 0 refers to differentiation with respect to the vector fields $x\partial_x$ and $x\partial_{y_i}$
- Second step:
  - Gain extra decay and apply L^2 Arzela-Ascoli