Introduction to Fourier Integral Operators - Lecture 2

Raluca Felea and Allan Greenleaf

Introductory Workshop

MSRI Microlocal Analysis Program

Overview of Lecture 2

- Conormal distributions (review)
- Nondegenerate phase functions
- Symplectic geometry, Lagrangians and canonical relations
- Fourier integral (Lagrangian) distributions
- Fourier integral operators (FIOs) and estimates
- Application: seismic imaging

Conormal distributions (review)

- $Y = \{x \in X^n : \phi_1(x) = \phi_2(x) = \cdots = \phi_k(x) = 0\}$, $\{d\phi_j\}_{j=1}^k$ lin ind

  $N^*Y = \{(x,\xi) \in T^*X : x \in Y, \xi = \Sigma_{j=1}^k \theta_j d\phi_j(x), \theta \in \mathbb{R}^k\}$

  $I^m(X;Y) = \left\{u(x) = \int_{\mathbb{R}^k} e^{i[\Sigma_{j=1}^k \theta_j \phi_j(x)]} \, a(x,\theta)\, d\theta,\ a \in S^m_{1,0}(R^n \times R^k)\right\}$

- $\mathrm{WF}(u) \subseteq N^*Y \setminus \mathbf{0}$

- Phase function $\phi(x,\theta) = \Sigma_{j=1}^k \theta_j \phi_j(x)$ on $X \times \mathbb{R}^k$ is **linear.**

Nondegenerate phase functions

- **Def.** $\phi(x,\theta)$ is a **phase function** on $X \times (\mathbb{R}^N \setminus 0)$ if it is smooth, $\mathbb{R}$-valued, homog of degree 1 in θ and

$$(d_x\phi,\, d_\theta\phi) \neq (0,0).$$

- THM. If $a \in S^m_{1,0}(X \times \mathbb{R}^N)$, then $u = \int_{\mathbb{R}^N} e^{i\phi(x,\theta)}\, a(x,\theta)\, d\theta \in \mathcal{D}'(X)$ then $WF(u) \subseteq \big\{(x, d_x\phi(x,\theta)) : d_\theta\phi(x,\theta) = 0\big\}$

- **Def.** ϕ is **nondegenerate** if $d_{x,\theta}(\frac{\partial\phi}{\partial\theta_1}), \ldots., d_{x,\theta}(\frac{\partial\phi}{\partial\theta_N})$ lin indep on

$$Crit_\phi := \big\{(x,\theta); d_\theta\phi = 0\big\} \subset X \times (\mathbb{R}^N \setminus 0).$$

Nondegenerate phase functions

- $\phi(x, \theta)$ nondegenerate $\iff$

$$rank\Big[d^2_{x\theta}\phi,\, d^2_{\theta\theta}\phi\Big] = N, \quad \forall (x,\theta) \in Crit_\phi$$

- **Prop 1.** ϕ nondeg $\implies$ $Crit_\phi$ is a closed, conic submfld of dim n. Furthermore, the map $h : Crit_\phi \to T^*X \setminus \mathbf{0}$,

$$h(x,\theta) = (x, d_x\phi(x,\theta)),$$

  is an immersion, and $h(Crit_\phi) = \Lambda_\phi =$ a conic **Lagrangian** submfld of $T^*X \setminus \mathbf{0}$ (to be defined).

- ϕ is said to **parametrize** Λ_ϕ.

Thm: WF of oscillatory integrals

Let $(x_0, \xi_0) \in T^*X \setminus \Lambda_\phi$, $\psi(x) \in \mathcal{D}(X)$ supported in nhood of x_0

- $\widehat{\psi u}(\xi) = \int \int e^{i(\phi(x,\theta) - x\cdot\xi)} a(x,\theta)\, \psi(x)\, d\theta\, dx$,
- Form vec fld near (x_0, ξ_0): $L = \frac{1}{|d_x\phi - \xi|^2} \sum_j (d_{x_j}\phi - \xi_j)\partial_{x_j}$

  $\implies L(e^{i(\phi(x,\theta) - x\cdot\xi)}) = e^{i(\phi(x,\theta) - x\cdot\xi)}$
- $|d_x\phi(x,\theta) - \xi| \geq c(|\xi| + |\theta|)$ on $\mathrm{supp}(a \cdot \psi)$, can integrate by parts

  $\implies \widehat{\psi u}$ rapidly decreasing on conic nhood of ξ_0
- Thus, $(x_0, \xi_0) \notin \mathrm{WF}(u)$.

Symplectic geometry: linear algebra

- **Def.** A **symplectic vector space** is a pair (V, ω), with ω a bilinear, nondegenerate, skew-symmetric form on V.
- If V is finite dim, dim(V) is necessarily even, say dim$(V) = 2n$.
- **Ex.** $V = \mathbb{R}^2$ with the area form $dx \wedge dy$
- **Ex.** $V = \mathbb{R}^{2n} = \{(x_1, \ldots, x_n, y_1, \ldots, y_n)\}, \quad \omega = \sum dy_j \wedge dx_j$
- $\omega((x, y); (x', y')) = \frac{1}{2} \sum (x_i y_i' - x_i' y_i)$

Symplectic geometry: linear algebra

- **Def.** Let (V, ω) be symplectic and $L \subseteq V$ a linear subsp. Then

  (i) $L^\omega := \{v \in V : \omega(u,v) = 0, \forall u \in L\}$,
  and $\dim(L^\omega) = \dim(V) - \dim(L)$ since ω nondegenerate

  (ii) L is **isotropic** if $L \subseteq L^\omega$, i.e., $\omega|_{L \times L} \equiv 0$

  (iii) L is **co-isotropic (involutive)** if $L^\omega \subseteq L$

  (iv) L is **Lagrangian** if $L = L^\omega$ ($\implies \dim(L) = \frac{1}{2}\dim(V)$)

- **Ex.** $\dim(L) = 1 \implies$ isotropic, $\text{codim}(L) = 1 \implies$ co-isotropic.

  Ex. $\dim(V) = 2 \implies$ any 1-dim subspace is Lagrangian

  Ex. In $\mathbb{R}^{2n}$, $L = \mathbb{R}^n \times \{0\}$ and $\{0\} \times \mathbb{R}^n$ are Lagrangian

Symplectic geometry: manifolds

- **Ex.** Cotangent bundle T^*X of a smooth X^n
- Local coordinates $(x_1, \ldots, x_n)$ on X, induce local coords $(x_1, \ldots x_n, u_1, \ldots u_n)$ on TX, $(x_1, x_2, \ldots x_n, \xi_1, \xi_2, \ldots \xi_n)$ on T^*X
- If $u \in T_xX$, then $u = \sum u_i \frac{\partial}{\partial x_i}$
- If $\xi \in T_x^*X$, then $\xi = \sum \xi_i dx_i$
- **Canonical 1-form** on T^*X: $\sigma := \xi dx = \sum_i \xi_i dx_i$, coord-indep.
- **Canonical 2-form:** $\omega := d\sigma = d\xi \wedge dx = \sum d\xi_i \wedge dx_i$, " "

  $t = (t_x, t_\xi),\ s = (s_x, s_\xi)$ then $\omega(s,t) = \frac{1}{2}(< t_\xi, s_x > - < s_\xi, t_x >)$

  ω is bilinear, skew-symmetric, nondegenerate, **closed**

Symplectic geometry: manifolds

- **Def.** (M, ω) is a **symplectic** manifold if ω is a closed diff 2-form on M and $\omega|_{T_x M}$ is symplectic for all $x \in M$. Thus, $\dim(M) = 2n$ and ω^n is a volume form orienting M.

  Ex. $(T^*\mathbb{R}^n, \sum d\xi_i \wedge dx_i)$, $(\mathbb{C}^n, \sum dy_i \wedge dx_i)$, (T^*X, ω)

- (T^*X, ω) has a bit more structure: it is **exact** (since $\omega = d\sigma$) and is **conic**, since there is a nice action of $\mathbb{R}_+$ on $T^*X \setminus \mathbf{0}$, $(x, \xi) \to (x, t\xi)$.

- **Def.** $\Gamma \subset T^*X \setminus \mathbf{0}$ is **conic** if $(x, \xi) \in \Gamma$ then $(x, t\xi) \in \Gamma, \forall t$.

  Ex. If $P(x, D) \in \Psi^m_{cl}(X)$ with principal symbol $p_m(x, \xi)$, then the **characteristic variety** of P,

$$\Sigma_P := \left\{(x, \xi) \in T^*X \setminus \mathbf{0} : p_m(x, \xi) = 0\right\} \text{ is closed, conic.}$$

Lagrangian manifolds

- **Def.** Let (M, ω) be a symplectic manifold and $L \subset M$ a smooth submanifold. Then L is **isotropic/co-isotropic/Lagrangian**, resp., if $T_x L \leq T_x M$ is isotropic/co-isotropic/Lagrangian, $\forall x \in L$.

  { Lagrangian submanflds } = { co-isotropic } $\cap$ { isotropic }
- **Prop.** L is Lagrangian iff $\omega|_L = 0$ and $\dim(L) = \frac{1}{2}\dim(M)$.
- **Ex.** If $f \in C^\infty_{\mathbb{R}}(X)$, then $\Lambda_f := \Big\{(x, df(x)) : x \in X\Big\} \subset (T^*X, \omega)$

  is Lagrangian, but not conic
- $\omega|_{\Lambda_f} = 0, \Longleftrightarrow f_{x_i x_j} = f_{x_j x_i}$
- zero-section $\mathbf{0} = \{(x, 0) : x \in X\}$ is Lagrangian, but not conic.
- Homogeneous microlocal analysis: work in $T^*X \setminus \mathbf{0}$.

Canonical transformation

- **Def.** If (M, ω_M) and (N, ω_N) be two symplectic manifolds of the same dimension, then a C^∞ map $\Phi : M \to N$ is a **canonical transformation** if $\Phi^* \omega_N = \omega_M$
- $\Phi^* \omega(V_1, \dots V_k) = \omega(\Phi(x))(D\Phi(V_1), \dots D\Phi(V_k))$

  Since ω_M nondeg, this $\implies$ Φ is a local diffeomorphism.

  Ex. A diffeom $\chi : X^n \to Y^n$ induces a canonical transformation, $\Phi : T^* X^n \longrightarrow T^* Y^n$,

$$\Phi(x, \xi) = \big(\chi(x),\, ((D\chi)^{-1})^t(\xi)\big)$$

- **Def** A **canonical graph** is the graph of a canonical transformation.

Conic Lagrangian manifolds

- **Prop.** If $Y \subset X$ is smooth, then its conormal bundle $N^*Y \setminus \mathbf{0}$ is a conic Lagrangian in $T^*X \setminus \mathbf{0}$.
- **Thm.** Any conic lagrangian Λ can be microlocally parametrized by a nondegenerate phase function ϕ. I.e., $\forall\, \lambda_0 = (x_0, \xi_0) \in \Lambda$, $\exists\, \phi$, a nondeg phase on a conic nhood of $(x_0, \theta_0) \in X \times (\mathbb{R}^{N_0} \setminus \mathbf{0})$, s.t. $\Lambda = \Lambda_\phi$ near λ_0.

  Sketch of pf. (i) If projection $(x, \xi) \to \xi$, is a submersion near λ_0, then microlocally Λ has form $\left\{(x, \xi) : x = \frac{\partial H}{\partial \xi}\right\}$, with $H(\xi)$ homog degree 1 and then $\phi = x \cdot \xi - H(\xi) \rightsquigarrow \Lambda$.
  (ii) Show (i) holds after a suitable quadratic change of coordinates.
- **Note:** A conic Lagrangian need not be of the form N^*Y for some smooth Y. For $H(\xi) = \frac{\xi_1^3}{\xi_2^2}$ above, Λ_ϕ is the closure of the conormal bundle of the smooth pts of the curve: $(\frac{x_1}{3})^3 = (\frac{x_2}{2})^2$.

Fourier integral distributions: Definition

- For $\phi(x,\theta)$ a nondeg phase, $Crit_\phi := \{(x,\theta); d_\theta\phi = 0\}$ and $h : Crit_\phi \to T^*X;\ h(x,\theta) := (x, d_x\phi(x,\theta))$, $h(Crit_\phi) = \Lambda_\phi = \{(x, d_x\phi); (x,\theta) \in C_\phi\}$
- **Prop.** Λ_ϕ is a conic Lagrangian submanifold. **Pf.** In fact, the canonical 1-form $\sigma = \xi \cdot dx = d_x\phi \cdot dx = d\phi - d_\theta\phi \cdot d\theta = 0$ on Λ_ϕ
- **Def.** The class $I^m(X;\Lambda) \subset \mathcal{D}'(X)$ of **Fourier integral distributions** of order m associated with Λ consists of all locally finite sums of

$$u_\phi = \int_{\mathbb{R}^{N_\phi}} e^{i\phi(x,\theta)}\, a(x,\theta)\, d\theta,\ a \in S_{1,0}^{m+\frac{dim X}{4}-\frac{N_\phi}{2}}(X \times \mathbb{R}^{N_\phi})$$

over all ϕ microlocally parametrizing $\Lambda_\phi \subseteq \Lambda$.

- **Recall:** $\mathrm{WF}(u_\phi) \subseteq \Lambda_\phi \subseteq \Lambda$

Fourier integral distributions: Examples

- **Ex.** Conormal distributions are Fourier integral distributions: For

$$Y = \{\phi_1(x) = \cdots = \phi_k(x) = 0\},$$

$$u(x) = \int e^{i\Sigma_{j=1}^k \theta_j \phi_j(x)}\, a(x,\theta)\, d\theta,$$

- $\phi(x,\theta) := \Sigma_{j=1}^k \theta_j \phi_j(x)$ is homog of deg 1

  $d\phi = \big((\phi_1(x), \ldots, \phi_k(x)), \sum \theta_j d\phi_j(x) \big) \neq (0,0)$

  is nondeg since $\{d(\phi_j(x))\}$ lin indep
- $\Lambda_\phi = \big\{(x, \sum \theta_i d\phi_j(x)) : x \in Y,\, \theta \in \mathbb{R}^k\big\} \setminus \mathbf{0} = N^*Y \setminus \mathbf{0}$

Fourier integral distributions: Examples

- $T_1 = \Psi$DO, Schwartz kernel: $K_{T_1}(x,y) = \int e^{i(x-y)\cdot\theta}\, a(x,y,\theta)\, d\theta$
- $T_2 =$ pull back by χ: $K_{T_2}(x,y) = \int e^{i(\chi(x)-y)\cdot\theta}\, a(x,y,\theta)\, d\theta$
- $T_3 =$ Radon transform: $K_{T_3}(\omega,s,y) = \int e^{i(y\cdot\omega - s)\theta}\, 1(s,y,\theta)\, d\theta$
- $T_4 =$ spherical mean operator: $K_{T_4}(x,y) = \int e^{i(|x-y|-t)\theta}\, 1(\theta)\, d\theta$
- $T_5 =$ Melrose-Taylor transf: $K_{T_5}(\omega,t,y,s) = \int e^{i((t-s-y\cdot\omega)\theta)}\, 1(\theta)\, d\theta$

Invariance of phase function

- **Thm.** For any two phase functions parametrizing the same Lagrangian, $\Lambda_\phi = \Lambda_{\tilde{\phi}}$, there is a chain of operations of the following types which transforms ϕ to $\tilde{\phi}$.
- 1) **Adding variables:** $\hat{\phi}(x,\theta,\sigma) := \phi + \frac{1}{2}\frac{\sigma^2}{|\theta|}$ a nondeg phase function
- 2) **Reducing variables:** stationary phase.
- 3) **Conic change of variables:** $\tilde{\theta}(x,\theta)$ and $\tilde{\phi} = \phi(x,\tilde{\theta}(x,\theta))$

$Crit_{\tilde{\phi}} = \{(x,\theta) : d_\theta\tilde{\theta} = 0\} = \{(x,\theta) : d_\theta\phi d_\theta\tilde{\theta} = 0\} = Crit_\phi$

$\Lambda_{\tilde{\phi}} = \{(x, d_x\tilde{\phi})\} = \{(x, d_x\phi + d_\theta\phi d_x\tilde{\phi})\} = \Lambda_\phi$

Then $\int e^{i\phi(x,\theta)}a(x,\theta)d\theta = \int e^{i\tilde{\phi}(x,\tilde{\theta})}\tilde{a}(x,\tilde{\theta})d\tilde{\theta}$.

Fourier integral operators (FIOs): Definition

- On $T^*(X \times Y)$ the natural symplectic form is $\omega_X + \omega_Y$

  But on $T^*X \times T^*Y$ the natural symplectic form is $\omega_X - \omega_Y$

- **Def.** A conic Lagrangian $C \subset \big((T^*X \setminus 0) \times (T^*Y \setminus 0), \, \omega_X - \omega_Y\big)$ is called a **canonical relation**.

  $C' := \{(x, \xi, y, -\eta)\}$ is then a Lagrangian w.r.t. $\omega_X + \omega_Y$, and *v.v.*

- **Def.** Let $C \subset (T^*X \setminus \mathbf{0}) \times (T^*Y \setminus \mathbf{0})$ be a closed, conic canonical relation. A **Fourier integral operator** associated to C is an operator $F : \mathcal{E}'(Y) \to \mathcal{D}'(X)$ whose Schwartz kernel $K_F \in I^m(X \times Y; C')$. The class of all such is denoted by $I^m(X, Y; C)$.

Fourier integral operators (FIOs)

- Thus, an FIO in $I^m(X,Y;C)$ is a locally finite sum of

$$Ff(x) = \int e^{i\phi(x,y,\theta)}\, a(x,y,\theta) f(y)\, d\theta dy,$$

  with $a \in S_{1,0}^{m+\frac{dimX+dimY}{4}-\frac{N}{2}}(X \times Y \times \mathbb{R}^N)$.

- $C = \Lambda_\phi = \{(x, d_x\phi; y, -d_y\phi) : d_\theta\phi = 0\}$
- **Ex.** If $X = Y, C = \Delta_{T^*X} = graph(Id_{T^*X})$, then

$$I^m(X,X;C) = \Psi^m(X).$$

Fourier integral operators: Examples

- $T_2 =$ pull back: $C = \{(x, \chi'(x)\theta; y, \theta) : \chi(x) = y\}$
- $T_3 =$ Radon transf: $C = \{(\omega, s, \theta y, -\theta; y, -\theta\omega) : s = y \cdot \omega\}$
- $T_4 =$ spherical mean operator:

$$C = \left\{ \big(x, \theta\frac{x-y}{|x-y|}; y, \theta\frac{x-y}{|x-y|}\big) : |x-y| = t,\, \theta \in \mathbb{R} \setminus 0 \right\}$$

- $T_5 =$ Melrose-Taylor tr: $C = \{(t, y, \theta, -\theta\omega; s, \omega, \theta, \theta y); t = s + y \cdot \omega\}$
- But: for $T_6 =$ Half-wave op. for $t \in \mathbb{R}$ fixed,

$$C = \{(x, \theta; y, \theta - d_y p(y, \theta)) : x - y + t d_\theta p(y, \theta) = 0\}$$

need not be conormal

Fourier integral operators: Projections

Geometry of canonical relations $\leftrightarrow$ structure of **projections**

- $$C \subset (T^*X \setminus 0) \times (T^*Y \setminus 0)$$
  $$\begin{array}{ccc} & \overset{\pi_L}{\swarrow} \qquad \overset{\pi_R}{\searrow} & \\ T^*X \setminus 0 & & T^*Y \setminus 0 \end{array}$$
- Note that $\mathrm{dim}(T^*X) = 2n_X$, $\mathrm{dim}(T^*Y) = 2n_Y$, $\mathrm{dim}(C) = n_X + n_Y$.
- **Prop.** At any point $c_0 \in C$, $\mathrm{corank}(\mathrm{D}\pi_L) = \mathrm{corank}(\mathrm{D}\pi_R)$.

Fourier integral operators: Projections

- **Def.** C is **nondegenerate** if one, hence both, projections have maximal rank everywhere.
- **Prop.** Suppose C is nondegenerate. (i) If dim X=dim Y, then both π_L, π_R are local diffeoms, and C is a **local canonical graph.**

  (ii) If $\dim(X) > \dim(Y)$, then π_L is an **immersion** and π_R is a **submersion.**

  We say that the **Bolker condition** is satisfied if, in addition, $\pi_L : C \to T^*X$ is globally injective.

Fourier integral operators: Radon transform

$\phi(\omega, s, y; \theta) = (y \cdot \omega - s)\theta$ on $(\mathbb{S}^{n-1} \times \mathbb{R} \times \mathbb{R}^n) \times (\mathbb{R} \setminus 0)$

$$\rightsquigarrow C_3 = \left\{(\omega, y \cdot \omega, \theta y, -\theta; y, -\theta\omega) : \omega \in \mathbb{S}^{n-1}, y \in \mathbb{R}^n, \theta \in \mathbb{R} \setminus 0\right\}$$

Coordinates on C_3 : $\omega \in \mathbb{S}^{n-1}$, $y \in \mathbb{R}^n$, $\theta \in \mathbb{R} \setminus 0$

- $\pi_L(y, \theta, \omega) = (\omega, y \cdot \omega, -\theta y, \theta)$,
- $\pi_R(y, \theta, \omega) = (y, -\theta\omega)$

$\mathrm{rank}(D\pi_R) = n + \mathrm{rank}\left(\frac{D(\eta)}{D(\omega, \theta)}\right) = 2n =$ maximal $\implies$

π_L, π_R diffeomorphisms, C_3 local canonical graph; π_L is 1-1, π_R is 2-1

Fourier integral operators: Spherical mean operator

- $C_4 = \{(x, \xi, x - t\frac{\xi}{|\xi|}, \xi) : x \in \mathbb{R}^n,\, \xi \in \mathbb{R}^n \setminus 0\}$
- $\pi_L(x, \xi) = (x, \xi)$,
- $\pi_R(x, \xi) = (x - t\frac{\xi}{|\xi|}, \xi)$
- $\pi_L,\ \pi_R$ are diffeomorphisms, C_4 is a canonical graph

Fourier integral operators: Solution of the wave eq

- $\phi(x,y,\theta,t) = (x-y)\cdot\theta + t|\theta|$, $x, y \in \mathbb{R}^n,\ t \in \mathbb{R}\setminus 0,\ \theta \in \mathbb{R}^n \setminus 0$
- $C = \{(x,t,\theta,|\theta|;y,\theta) : x - y + t\frac{\theta}{|\theta|} = 0, t \neq 0\}$
- $\pi_L(x,t,\theta) = (x,t,\theta,|\theta|)$
- $\pi_R(x,t,\theta) = (x - y + t\frac{\theta}{|\theta|}, \theta)$
- max rank, π_L a immersion, 1-1, π_R a submersion

Estimates for nondegenerate FIOs

- **Thm. (Hörmander)** If $C \subset (T^*X \setminus \mathbf{0}) \times (T^*Y \setminus \mathbf{0})$ is a nondegenerate canonical relation and $F \in I^{m - \frac{|d_X - d_Y|}{4}}(X, Y; C)$, then $F : H^s_{comp}(Y) \to H^{s-m}_{loc}(X)$.

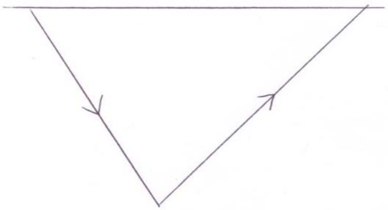
- Radon transf: $T_3 \in I^{-\frac{n-1}{2}}(C_3) \implies T_3 : H^s_{comp} \to H^{s+\frac{n-1}{2}}_{loc}$

- Spher means: $T_4 \in I^{-\frac{n-1}{2}}(C_4) \implies T_4 : H^s_{comp} \to H^{s+\frac{n-1}{2}}_{loc}$

- Soln of WE: $T \in I^{-\frac{1}{4}}(C), T : H^s_{comp} \to H^s_{loc}$

Inverse problems in seismology

- surface (source) pressure field: data (receiver)



$c(x_1, x_2, x_3)$: subsurface (image)

- F : image $\rightarrow$ data: forward operator
- Wave equation:
  $(*)\ \frac{1}{c^2(x)} \frac{\partial^2 p}{\partial t^2}(x,t) - \triangle p(x,t) = \delta(t)\delta(x-s)$
  $$p(x,t) = 0, t < 0$$
- $p(x,t)$ is the pressure field resulting from a pulse at the source s
- $c(x)$ = velocity field independent of direction

Formal linearization

- $c = c_0 + \delta c$
- $p = p_0 + \delta p$
- $(**)\ \frac{1}{c_0^2(x)} \frac{\partial^2(\delta p)}{\partial t^2}(x,t) - \triangle(\delta p)(x,t) = \frac{\partial^2 p_0}{\partial t^2} \frac{2(\delta c)}{c_0^3}$
- $\delta p = 0, t < 0$
- $F : \delta c \longrightarrow \delta p|_{\Sigma \times (0,T)}$

Analysis

- Rakesh: F is an FIO
- to find the image we use F^*
- Under the travel time injectivity cond (Bolker cond), F^*F is a ΨDO.
- If background ray geometry has caustics of at worst fold type (map $(s,t) \to x(s,t)$ has fold singularities, single source $\implies$
- C is a two sided fold and $C^t \circ C = \Delta \cup C_1$ where C_1 is another two sided fold (Melrose-Taylor)
- **Thm. (F. - Nolan)** If $F \in I^m(C)$ then $F^*F \in I^{2m,0}(\Delta, C_1)$.