Overview

1. Symbol calculus
2. Functional and composition calculus
3. Examples and applications
4. Extensions and generalizations of FIO calculus
5. Readings for all three lectures
Symbol calculus of Fourier integral distributions

For $\Lambda \subset T^* X^n \setminus \mathbf{0}$ a smooth conic Lagrangian,

$$I^m(X; \Lambda) = \text{all locally finite sums of } u \in \mathcal{D}'(X)$$

given by oscillatory integrals

$$u = u(a, \phi) := \int_{\mathbb{R}^N} e^{i\phi(x, \theta)} a(x, \theta) \, d\theta, \quad a \in S^{m-\frac{N}{2} + \frac{n}{4}}_{1,0}$$

with $\phi(x, \theta)$ a nondegenerate phase on $X^n \times (\mathbb{R}^N \setminus \mathbf{0})$

$$\sim \text{Crit}_\phi := \{(x, \theta) : d_\theta \phi(x, \theta) = 0\}$$

$$\sim \Lambda_\phi := \{(x, dx \phi) : (x, \theta) \in \text{Crit}_\phi\} \subset \Lambda.$$
Symbol calculus of Fourier integral distributions

- Define $n$-form $\mu_\phi$ on $\text{Crit}_\phi$ by requiring

$$\mu_\phi \wedge d\left(\frac{\partial \phi}{\partial \theta_1}\right) \cdot \cdot \cdot \wedge d\left(\frac{\partial \phi}{\partial \theta_N}\right) = dx_1 \cdot \cdot \cdot \wedge dx_n \wedge d\theta_1 \cdot \cdot \cdot \wedge \theta_N$$

- If $\lambda_i$ are local coord on $\text{Crit}_\phi$ then $\mu_\phi = fd\lambda_1 \cdot \cdot \cdot \wedge d\lambda_n$, with

$$f = \frac{dx_1 \cdot \cdot \cdot \wedge dx_n \wedge d\theta_1 \cdot \cdot \cdot \wedge \theta_N}{d\lambda_1 \cdot \cdot \cdot \wedge d\lambda_n \wedge d\left(\frac{\partial \phi}{\partial \theta_1}\right) \cdot \cdot \cdot \wedge d\left(\frac{\partial \phi}{\partial \theta_n}\right)}$$

- To obtain an invariantly defined principal symbol, $\sigma_{\text{prin}}(u)$, if

$$a^0 := \left[ a \big|_{\text{Crit}_\phi} \right] \in S_{1,0}^{m-\frac{N}{2} + \frac{n}{4}} / S_{1,0}^{m-\frac{N}{2} + \frac{n}{4} - 1},$$

**Def.** The principal symbol $\sigma_{\text{prin}}(u)$ of $u(a, \phi)$ is the push-forward of the half-density $a^0 \sqrt{\mu_\phi}$ from $\text{Crit}_\phi$ to $\Lambda$. 
Suppose $A \in \text{dom}(C; X, Y)$. What about (formal) $A^*$? If

$$K_A(x, y) = \int_{\mathbb{R}^N} e^{i\phi(x, y, \theta)} a(x, y, \theta) d\theta, \quad a \in S^{m-\frac{N}{2} + \frac{nX+nY}{4}},$$

then

$$K_{A^*}(y, x) = \overline{K_A(x, y)}$$

$$= \int_{\mathbb{R}^N} e^{-i\phi(x, y, \theta)} \overline{a}(x, y, \theta) d\theta, \quad \overline{a} \in S^{m-\frac{N}{2} + \frac{nX+nY}{4}}$$

$$\implies A^* \in \text{dom}(C^t; Y, X), \text{ where } C^t \text{ is the transpose relation.}$$
Suppose $A_1 \in I^{m_1}(C_1; X, Y)$, $A_2 \in I^{m_2}(C_2; Y, Z)$ are properly supported.

- **Q.** Is $A_1 A_2$ an FIO? **No** in general, but **yes** if we impose some geometric conditions.

- **Note**

\[
WF_{A_1 A_2} \subseteq WF_{A_1} \circ WF_{A_2} = W_K(K_{A_1})' \circ WF(K_{A_2})' \\
\subseteq C_1 \circ C_2 \subset (T^*X \setminus 0) \times (T^*Z \setminus 0)
\]

Basic examples show $C_1 \circ C_2$ need not be a smooth canonical relation. However, under a **transversality** or **clean intersection** condition, it is, and the operator theory follows the geometry.
Transverse intersection

- **Def.** $S_1, S_2 \subset M$ intersect **transversally** if $T_m S_1 + T_m S_2 = T_m M$ for all $m \in S_1 \cap S_2$. (Holds $\iff N^*_m S_1 \cap N^*_m S_2 = (0)$.) Write $S_1 \pitchfork S_2$.

- **Prop.** If $S_1 \pitchfork S_2$, then
  
  (i) $S_3 := S_1 \cap S_2$ is smooth;
  
  (ii) $\text{codim}(S_3) = \text{codim}(S_1) + \text{codim}(S_2)$; and
  
  (iii) $T S_3 = T S_1 \cap T S_2$ at all points.

- **Ex.** In $\mathbb{R}^3$: $\{z = 0\} \pitchfork \{z = x\}$, but $\{z = 0\} \not\pitchfork \{z = xy\}$. 
Transverse intersection

- For $C_1 \subset (T^*X \setminus 0) \times (T^*Y \setminus 0)$ and $C_2 \subset (T^*Y \setminus 0) \times (T^*Z \setminus 0)$,

$$C_1 \circ C_2 = \{ (x, \xi, z, \zeta) : \exists (y, \eta) \text{ s.t. } (x, \xi, y, \eta) \in C_1, (y, \eta, z, \zeta) \in C_2 \}$$

$$= (\pi_1 \times \pi_4)((C_1 \times C_2) \cap (T^*X \times \Delta_{T^*Y} \times T^*Z))$$

- To have a chance of $A_1 A_2$ being an FIO associated with a smooth canonical relation, need that the intersection set be smooth.

- One way to get this is to demand that the intersection be transverse.
Transverse intersection calculus

- **Thm. (Hörmander)** Suppose
  \[ A_1 \in I^{m_1}(C_1; X, Y), \ A_2 \in I^{m_2}(C_2; Y, Z) \] are properly supported. If
  \[ C_1 \times C_2 \] intersects \[ T^*X \times \Delta_{T^*Y} \times T^*Z \] transversally, then \( C_1 \circ C_2 \) is a smooth canonical relation and
  \[ A_1 A_2 \in I^{m_1+m_2}(C_1 \circ C_2; X, Z) \]

- If either \( C_1 \) or \( C_2 \) is a local canonical graph, then \( A_1 A_2 \) is covered by the \( \hbar \) calculus.

- In particular, \( I^m(C; X, Y) \) is closed under composition on the right with \( \Psi^0(Y) \) and on the left with \( \Psi^0(X) \).

- If \( C \) is a canonical graph and \( A \in I^m(C; X, Y) \) is properly supported, then \( A^* A \in \Psi^{2m}(Y) \), and \( A \) elliptic at \( (x_0, \xi_0, y_0, \eta_0) \) \( \implies \) \( A^* A \) elliptic at \( (y_0, \eta_0) \).
**Clean intersection calculus**

- **Def.** $S_1, S_2 \subset M$ intersect **cleanly** if (i) $S_3 := S_1 \cap S_2$ is smooth; and $TS_3 = TS_1 \cap TS_2$ at all points. The **excess** of the intersection is $e := \text{codim}(S_1) + \text{codim}(S_2) - \text{codim}(S_3) \geq 0$.

**Ex.** $S_1 = x$-axis and $S_2 = y$-axis in $\mathbb{R}^3$, with excess $e = 2 + 2 - 3 = 1$.

**Ex.** $S_1 = x$-axis and $S_2 = \{ y = x^2 \}$ do not intersect cleanly in $\mathbb{R}^2$.

- **Thm. (Duistermaat-Guillemin; Weinstein)** If $C_1 \times C_2$ intersects $T^*X \times \Delta_{T^*Y} \times T^*Z$ cleanly with excess $e$, then $C_1 \circ C_2$ is smooth and

$$A_1 A_2 \in I^{m_1 + m_2 + \frac{e}{2}}(C_1 \circ C_2; X, Z).$$
Clean intersection calculus: flowouts

**Ex.** Let $\Sigma \subset T^*X^n \setminus 0$ be a conic hypersurface.

- $\Sigma$ is automatically **co-isotropic**: $(T\Sigma)\omega \subset T\Sigma$ at all pts.
- Microlocally, can write $\Sigma = \{p(x, \xi) = 0\}$, $p \in C^\infty_{\mathbb{R}}$, homog of deg 1.
- $(T\Sigma)\omega = \mathbb{R} \cdot H_p$, where $H_p$ is the **Hamiltonian vector field** of $p$,

$$
H_p(x, \xi) = (dp(x, \xi))\omega = \sum_{j=1}^{n} \frac{\partial p}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial p}{\partial x_j} \frac{\partial}{\partial \xi_j}
$$

- But $H_p \in T\Sigma$, since $\langle dp, H_p \rangle = \omega(H_p, H_p) = 0$ by skew-symmetry.
- Thus, $\Sigma$ is foliated by the integral curves of $H_p$, called the **bicharacteristic curves** of $\Sigma$, which are **nonradial** if $H_p \parallel \xi \cdot \partial \xi$.

The curve passing through $(x, \xi) \in \Sigma$ is denoted $\Xi_{x, \xi}$. 


Clean intersection calculus: flowouts

- **Def.** The *flowout relation* of $\Sigma$,

$$C_\Sigma = \{(x, \xi, y, \eta) : (x, \xi) \in \Sigma, (y, \eta) \in \Xi_{x,\xi}\} \subset (T^*X \setminus 0) \times (T^*X \setminus 0)$$

is a smooth, conic canonical relation.

**Note:** $C_\Sigma$ is degenerate. $D\pi_L$, $D\pi_R$ drop rank by 1 everywhere.

- $C_\Sigma \circ C_\Sigma$ covered by the clean intersection calc, with excess $e = 1$:

$$I^{m_1}(C; X, X) \circ I^{m_2}(C; X, X) \subseteq I^{m_1+m_2+\frac{1}{2}}(C; X, X)$$

- Results in a loss of $1/2$ derivs on $L^2$-based Sobolev spaces:

**Thm.** $I^{m}(C; X, X) : H^s_{\text{comp}}(X) \to H^{s-m-\frac{1}{2}}_{\text{loc}}(X)$.
Clean intersection calculus: flowouts

- Flowout relations $C_\Sigma$ describe the propagation of singularities of solutions to $Pu = f$, where $P(x, D) \in \Psi^m_{cl}(X)$.

- **Def.** $P(x, D) \in \Psi_{cl}$ is of **real principal type** if $p(x, \xi) := \sigma_{prin}(P)$ is $\mathbb{R}$-valued, $d_{x,\xi}p \neq (0, 0)$ at $\Sigma = p^{-1}(0)$, and no bicharacteristic $\Xi_{x,\xi}$ of $p$ is trapped over a compact set $K \subset X$. (In particular, there are no radial points.)

- **Thm. (Duistermaat-Hörmander)** If $P(x, D)$ is RPT and $Pu = f$, then $WF(u) \setminus WF(f)$ is a union of maximally extended $\Xi_{x,\xi}$. Furthermore, there exists a two-sided parametrix $Q$, $QP = I - R_1$ and $PQ = I - R_2$ with $R_1, R_2 \in \Psi^{-\infty}(X)$, with $Q \in I^{\frac{1}{2}}_{-m}(C_\Sigma)$ away from $\Delta_{T^*X}$. 
Applications: Egorov’s Theorem

- Let $\Phi : T^*Y \setminus 0 \to T^*X \setminus 0$ be a canonical transformation defined on a conic nhood of $(y_0, \eta_0)$. Then $C := graph(\Phi)$ is a canonical graph.

- Let $F \in I^0(C; X, Y)$ be an elliptic FIO, and $G \in I^0(C^t; Y, X)$ a parametrix (microlocal inverse mod $C^\infty$), with $C^t = graph(\Phi^{-1})$:

$$GF \equiv I \text{ and } FG \equiv I \mod C^\infty.$$

- **Thm. (Egorov)** If $P(x, D) \in \Psi^m(X)$, then $FPG \in \Psi^m(Y)$, with

$$\sigma_{prin}(FPG)(y, \eta) = \sigma_{prin}(P)(\Phi(y, \eta))$$

- $\implies$ Large literature on reducing $\Psi$DO to normal forms, proving propagation of singularities or local solvability.
Suppose $Z \subset X^n \times Y^n$, codim $k$. Consider

$$\pi_X \quad Z \quad \pi_Y$$

$$\leftarrow \quad \rightarrow \quad X \quad Y$$

- **Def.** $Z$ is a **double fibration** if $\pi_X : Z \to X$ and $\pi_Y : Z \to Y$ are submersions. Then, $\forall x \in X, y \in Y$,

$$Y_x := \pi_Y \pi_X^{-1}(\{x\}) \subset Y$$ and $$X^y := \pi_X \pi_Y^{-1}(\{y\}) \subset X$$ are codim $k$

- Choice of smooth densities on $X, Y, Z$ induces pair of **generalized Radon transforms**, $\mathcal{R} : \mathcal{E}'(Y) \to \mathcal{D}'(X)$ and $\mathcal{R}^t : \mathcal{E}'(X) \to \mathcal{D}'(Y)$,

$$\mathcal{R} f(x) = \int_{Y_x} f(y) \, dy \quad \text{and} \quad \mathcal{R}^t g(y) = \int_{X^y} f(x) \, dx.$$
Applications: Generalized Radon transforms

$Z$ is the **incidence relation** of a generalized Radon transform, $\mathcal{R}$.

- **Guillemin-Sternberg:** Schwartz kernel of $\mathcal{R} = \delta_Z$, which is a conormal, hence Fourier integral distribution: Locally describe $Z$ as

  $$Z = \{ (x, y) : \Phi_1(x, y) = \cdots = \Phi_k(x, y) = 0 \}.$$  

- Writing $\delta_Z$ as shorthand for a smooth multiple of $\delta_{\mathcal{R}^k}(\Phi)$,

  $$\delta_Z(x, y) = \int_{\mathbb{R}^k} e^{i \sum_{j=1}^k \theta_j \Phi_j(x, y)} a(x, y) d\theta, \quad a \in S_{1,0}^0(X \times Y \times \mathbb{R}^k)$$

  $$\implies \quad \mathcal{R} \in I^{0 + \frac{k}{2} - \frac{n_X + n_Y}{4}}(C; X, Y),$$

  where

  $$C = N^*Z' \subset (T^*X \setminus 0) \times (T^*Y \setminus 0)$$

  and $\mathcal{R}^t \in I^{\frac{n_X + n_Y - 2k}{4}}(C^t; Y, X)$.  


Applications: Generalized Radon transforms

- If $C$ is a canonical graph, then $R^*R \in \Psi^{nX+nY-2k/2}(Y)$, elliptic if $R$ is.

- $\exists$ parametrix $Q \in \Psi^{-nX+nY-2k/2}(Y)$, $QR^*R \equiv I \mod C^\infty$, and thus $Rf$ determines $f \mod C^\infty$, $\forall f \in E'(Y)$.

- **Ex.** Radon transform: $Y = \mathbb{R}^n$, $X = S^{n-1} \times \mathbb{R}$,

  \[ Z = \{ (\omega, s, y) : s - \omega \cdot x = 0 \}. \]

  \[ R^*Rf = c_n f * |y|^{1-n}, \text{ which has inverse } c_n (-\Delta)^{n-1/2}. \]

  The **filtered backprojection** inversion formulae for the Radon transform,

  \[ f = c_n ((-\Delta)^{n-1/2} R^*) Rf = c_n R^*(|\partial_s|^{n-1}) Rf, \]

  thus generalize (mod $C^\infty$) to a wide variety of GRTs.
Applications: Generalized Radon transforms

- Suppose $n_X > n_Y$ (R is **overdetermined**).
  \[
  \dim(T^*X) = 2n_X > \dim(C) = n_X + n_Y > \dim(T^*Y) = 2n_Y.
  \]

- Then $C = N^*Z'$ is nondegen, i.e., $\pi_L : C \to T^*X$ has maximal rank, iff $D\pi_L$ is injective.

- Clean intersection calculus applies to $R^*R$, with excess $e = n_X - n_Y$, but to make sure that $R^*R$ is only a $\Psi$DO, need $C^t \circ C \subseteq \Delta_{T^*Y}$.

- **Def. (Guillemin)** $R$ (or $Z$ or $C$) satisfies the **Bolker condition** if, in addition to $D\pi_L : TC \to T(T^*X)$ being injective, the map $\pi_L : C \to T^*X$ is injective. I.e., not only is $\pi_L$ infinitesimally 1-1, it is globally 1-1. (Makes sense for general canonical relations.)
Applications: Generalized Radon transforms

- **Thm. (Guillemin-Sternberg)** Suppose $C \subset (T^*X \setminus 0) \times (T^*Y \setminus 0)$ is a canonical relation satisfying the Bolker condition, and

  $$F \in I^{m-\frac{nX-nY}{4}}(C; X, Y)$$

  is elliptic and properly supported. Then $F^* F \in \Psi^{2m}(Y)$, +elliptic.

  Hence, $u$ is determined mod $C^\infty(Y)$ by $F u$ mod $C^\infty(X)$, $\forall u \in \mathcal{E}'(Y)$.

- **Ex.** $k$-plane transform on $\mathbb{R}^n$: $\mathcal{R}_{k,n} \in I^{-\frac{k}{2}} - \frac{(k+1)(n-k)-n}{4} (C; M_{k,n}, \mathbb{R}^n)$

- **Ex.** $(M^n, g)$ a Riemannian manifold without conjugate points has a $(2n - 2)$-dimensional space $\mathcal{G}$ of geodesics. The **X-ray transform** on $M$, defined by $X f(\gamma) = \int_\gamma f \, ds$, satisfies the Bolker condition, $X^* X \in \Psi^{-1}(M)$, and $X f \mod C^\infty$ determines $f \mod C^\infty$. 
An example where Bolker is violated

- X-ray transform on \((M, g) = (S^n, g_0)\)

- \(X \in I^{-\frac{1}{2} - \frac{(2n-2)-n}{4}} (C; \mathcal{G}, M)\) with \(C \subset (T^* \mathcal{G} \setminus 0) \times (T^* M \setminus 0)\) is nondeg., but \(\pi_L : C \rightarrow T^* \mathcal{G}\) is 2-1.

- Composition \(X^* X\) is covered by clean intersection calc, but

\[
X^* X \in I^{-1}(\Delta) \cup I^{-1}(\Gamma)
\]

where \(\Gamma\) is the graph of the canonical transf induced by antipodal map, and \(X\) has a large kernel (all odd distributions).
Paired Lagrangian distributions

- ∃ need for distributions [operators] whose wavefront sets [relations] are not a smooth Lagrangian [canonical relation]:

- Duistermaat-Hörmander constructed parametrices $Q$ for RPT operators $P(x, D)$ have

$$WF_Q \subseteq \Delta_{T^*X} \cup C_\Sigma$$

where $C_\Sigma$ is the flowout of $\Sigma$. $\Delta \cap C_\Sigma$ cleanly with excess $e = n - 1$.

- Each of $\Delta_{T^*X}$, $C_\Sigma$ is smooth, but their union is not, and $K_Q$ is not simply a sum in $I^{m_1}(\Delta) + I^{m_2}(C_\Sigma)$. 
Paired Lagrangian distributions

- Melrose-Uhlmann-Guillemin-Mendoza introduced classes of Lagrangian-like distributions associated with pairs $\Lambda_0, \Lambda_1 \subset T^*X \setminus 0$ which intersect cleanly in codimension $k = 1, 2, \ldots$. Denoted

$$I^{p,l}(X; \Lambda_0, \Lambda_1), \quad p, l \in \mathbb{R}.$$

- Just as $u \in I^m(\Lambda)$ can be characterized either as oscillatory integrals or in terms of iterated regularity, $I^{p,l}$ can be characterized either as

(i) oscillatory integrals with certain types of product type symbols; or

(ii) distributions satisfying iterated regularity with respect to $P_j \in \Psi^1_{cl}$ with $\sigma_{prin}$ vanishing on $\Lambda_0 \cup \Lambda_1$.

- If $u \in I^{p,l}(\Lambda_0, \Lambda_1)$ then microlocally away from $\Lambda_0 \cap \Lambda_1$,

$$u \in I^{p+l}(\Lambda_0 \setminus \Lambda_1) \text{ and } u \in I^{p}(\Lambda_1 \setminus \Lambda_0).$$
Paired Lagrangian operators

- If $C_0, C_1 \subset (T^*X \setminus 0) \times (T^*Y \setminus 0)$ are a cleanly intersecting pair, then

$$I^{p,l}(C_0, C_1; X, Y) = \text{operators } T \text{ with } K_T \in I^{p,l}(C'_0, C'_1).$$

- When $Y = X, C_0 = \Delta_{T^*X}$: “ΨDO with singular symbols”.

- $I^{p,l}$-operators arise in several applications:
  (i) Parametrices for RPT: $Q \in I^{1/2-m, -1/2} (\Delta, C_\Sigma)$ [Melrose-Uhlmann]
  (ii) Parametrices for restricted X-ray transforms [G.- Uhlmann];
  (iii) Linearized inverse prosbs for seismic, radar imaging [Nolan, Felea];
  (iv) Composing FIOs outside the clean intersection calculus [G.-Uhlmann, Felea].
Paired Lagrangian operators

- **Ex.** \( x = (x', x'') \), \( C_0 = \Delta_{T^* \mathbb{R}^n} \), \( C_1 = N^* \{ x' - y' = 0 \} \)

- **Def. 1.** \( K(x, y) = \int e^{i((x' - y') \cdot \xi' + (x'' - y'') \cdot \xi'')} a(x, \xi) d\xi' d\xi'' \),

\[ |\partial_x^\alpha \partial_{\xi'}^\beta \partial_{\xi''}^\gamma a| \leq c_{\alpha \beta \gamma} (1 + |\xi'| + |\xi''|)^{m-|\beta|} (1 + |\xi''|)^{m'-|\gamma|} \]

- **Def. 2.** Iterated regularity: \( u \in I^{p, l}(C_0, C_1) \) if \( P_1 P_2 \ldots P_N K \in H_{loc}^{s_0} \)
where \( P_j \in \Psi^1_{cl} \) with \( \sigma(P_j) \) vanishing on \( C_0 \cup C_1 \)
Beyond the standard FIO calculus

- **Recall:** Melrose-Taylor Radon transform ($T_5$ from Lec. 1), $R_{MT} : \mathcal{D}'(\partial\Omega \times \mathbb{R}) \to \mathcal{D}'(S^{n-1} \times \mathbb{R})$, given by

$$R_{MT}(f)(\omega, t) = R_{MT}(f)(\omega, t) := \int \int \{y \cdot \omega = t-s\} \subset \partial\Omega \times \mathbb{R} f(y, s)$$

- $R_{MT} \in I^{-(n-1)/2}(C)$, with $C$ not a canonical graph. Both $\pi_L, \pi_R$ have degeneracies of Whitney fold type. Such $C$ called **folding canonical relations**. $T \in I(C)$ lose $1/6$ deriv on $L^2$.

- M-T already observed that the composition $C^t \circ C$ is not a smooth canonical relation, but $\subset \Delta_{T^*}(\partial\Omega \times \mathbb{R}) \cup C_1$, where $C_1$ intersects $\Delta$ cleanly in codim 1.

- **Thm. (Nolan-Felea).** If $C$ is a folding canonical relation and $F \in I^m(C; X, Y)$ then $F^* F \in I^{2m,0}(\Delta_{T^*Y}, C_1)$. 
Basic references: Classic articles


- Older papers of historical interest for introducing important ideas: V. Maslov, Y. Egorov, ...
Basic references: Books


Symplectic geometry: Books


Beyond the standard FIO calculus


Paired Lagrangian distributions and operators


Semiclassical FIOs
