\[ \mathbb{R}_x \times S^1_{\theta} \]
\[ \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \theta^2} \]
\[ V \in L^1_c (\mathbb{R}; \mathbb{C}) \]
\[ -\Delta + V \text{ on } \mathbb{R} \]

one-dim Schrödinger operator
\[ -\frac{d^2}{dx^2} + V_0 \]
\[ V_0(x) = \frac{1}{2\pi} \int_{2\pi} v(x, \theta) d\theta \]

Spectrum \(-\Delta + V\) on \(\mathbb{R}\)

continuous spectrum \([0, \infty)\)

\[ 0 \quad 1 \quad 4 \quad 9 \]

up to infinitely many eigenvalues.

only accumulation point at infinity
Resolvent: \( R_v(z) = (-\Delta + V - z^2)^{-1} \), if \( \text{Im} \ z > 0 \)

If \( x \in C_0^\infty(X) \), then \( xR_v(z)x \) has a meromorphic continuation to \( \mathbb{C} \): smallest Riemann surface on which

\[ T_j(z) := (z^2 - j^2)^{1/2} \]

is a single-valued analytic function for all \( j \in \mathbb{N} \). (\( \text{Im} \ z > 0 \))

(\( \text{Im} \ T_j(z) > 0 \) if \( z \) in the physical space \( \mathcal{H} \))

\( R_v(z) \) is bounded on \( L^2(X) \).

Poles of \( xR_v(z)x \) are resonances.

1-D problem: \[-\frac{d^2}{dx^2} + W, \ \omega \in L_c^0(\mathbb{R})

\[ R_{w,0}(\lambda) = (-\frac{d^2}{dx^2} + W - \lambda^2)^{-1}, \ \text{if} \ \text{Im} \ \lambda > 0 \]

meromorphic continuation to \( \mathbb{C} \).
\[ R_{v_0} (S) = \sum_{j=0}^{\infty} R_{v_0} (T_j(S)) P_j^\perp j > 0 \]

\text{projection onto span e}^{ij\theta} e^{-ij\theta}

Cartoon of partition of \( \hat{\mathbb{Z}} \)

\text{physical spaces}

\[ z = T e(S) \text{ - give complex structure} \]

\[ \text{nbhd of } e^{\pm \theta} \text{ threshold} \]

Set \[ U_m(x) = \frac{1}{2\pi} \int V(x, \theta) e^{-i m \theta} \, d\theta \quad m \in \mathbb{Z} \]
Theorem: Let $V \in L^r_c(X)$, $\|V\|_{L^r} = O(m^{-\frac{1}{2}})$
Suppose $\lambda_0 \in \mathbb{C}$ is a pole of $R_{\lambda_0}(\lambda)$ with
multiplicity $M_{\lambda_0}(\lambda_0)$. Then there is a $C > 0$ so that
for $l$ sufficiently large there are exactly $2M_{\lambda_0}(\lambda_0)$
poles of $R_{l}(S')$, when counted with multiplicity
in $\{S \in \mathbb{B}_e(l \lambda_0 + l) : |T_l(S) - \lambda_0| < C\}$

Theorem: $V$ be as above. Given $P > 0$, set
$$\Delta_P = \{\lambda_j \in \mathbb{C} : \lambda_j \text{ is a pole of } R_{\lambda_0}(\lambda), |\lambda_j| \leq P + 1\}.$$ 
Then there is a $C > 0$ so that for $l$ sufficiently large
there are no poles of $R_{l}(S)$ in
$$\{S \in \mathbb{B}_e(P) : |T_l(S) - \lambda_j| > C e^{-\frac{1}{2}M_{\lambda_0}(\lambda_j)}$$
for all $\lambda_j \in \Delta_P$.

If $V \in C^\infty$, can improve to
$$|T_l(S) - \lambda_j| > C l^{-\frac{3}{2}} M_{\lambda_0}(\lambda_j)$$
Thm: Let $V \in C^\infty_0(X)$, & let $\lambda_0$ is a simple pole of $R_{v_0}(\lambda)$, and that

$$R_{v_0}(\lambda) = \frac{i}{\lambda - \lambda_0} u \otimes u$$

is analytic near $\lambda - \lambda_0$.

Then for $t$ suff large, $R_v(t)$ has

either two simple poles $\pm \xi_0$ in $B_1(\xi_0) + (t)$ satisfying

$$T_{1}(\xi_0) = \lambda_0 - \frac{1}{4t} \sum_{k=0}^{\infty} \frac{1}{k^2} \int (k^2 V \cdot V_k + V_k \cdot V_k') u^2 \, d\lambda + O(t^{-3})$$

or a simple pole of multiplicity 2 with same asympt expansion.
Cor: Suppose $V \in C^\infty_c(X; \mathbb{R})$. Suppose for each $\rho > 0$ there is a sequence $\{\ell_j\} = \ell_j(\rho) \subset \mathbb{N}$, $\ell_j \to \infty$ as $j \to +\infty$, so that $-\Delta$ and $-\Delta + V$ have the same resonances in $B_{\ell_j}(\rho)$.

Then $V \equiv 0$.

Steps: 1) $V \equiv 0$ by showing $-\frac{d^2}{dx^2} + V_0$ has resonance only at 0.

2) Correction term, $V - \kappa = \overline{V}_\kappa$
Wave equation:
\[(\partial_t^2 - \Delta + V)u = 0 \text{ on } X \times (0, \infty),\]
\[(u, u_t)|_{t=0} \in C^\infty_c(X) \times C^\infty_c(X)\]

Thm: \(V \in C^\infty_c(X, \mathbb{R})\). Suppose \(-\frac{d^2}{dx^2} + V_0\) on \(\mathbb{R}\) has no negative eigenvalues & does not have a resonance at 0. For \(k_0 \in \mathbb{N}\), can write
\[
U(t) = U_{ev}(t) + U_{thr}(t) + U_r(t)
\]
where \(U_{ev}\) contribution of eigenvalues of \(-\frac{d^2}{dx^2} + V_0\),
\[
U_{thr}(t) = b_{00} + \sum_{K=0}^{k_0-1} t^{K-\frac{1}{2}} \sum_{j=1}^{\infty} \left( e^{it_j} b_{jk}^+ + e^{-it_j} b_{jk}^- \right)
\]
If \(\chi \in C^\infty_c(X)\), \(m \in \mathbb{N}\),
\[
\sum_j \| \chi b_{jk}^+ \|_{H^m} < \nu
\]
\[
\| \chi U_r \|_{H^m} \leq C t^{-k_0 - \frac{1}{2}}
\]
The last thm follows rest + work w/K. Datchev.

Sources of inspiration:

* paper of Drouot

resonances of $-\Delta + W_\varepsilon$ on $\mathbb{R}^d$, $d$ odd

$$W_\varepsilon(x) = V_0(x) + \sum_{k \in \mathbb{Z}} V_k(x)e^{ik \cdot x/\varepsilon} \quad \varepsilon > 0 \text{ small}$$

as $\varepsilon \downarrow 0$, resonances of $-\Delta + W_\varepsilon$ well-approximated by resonances of $-\Delta + V_0$.

* paper on eigenvalues of $-\Delta + V$ on $\mathbb{S}^d$

Weinstein (Guillemin, Widim, Friedlander …)