EMILY PETERS: SUBFACTORS AND PLANAR ALGEBRAS, II

Last time we dove into planar algebras; today we'll get to subfactors too.

Definition 0.1. A *von Neumann algebra* A is a unital, $*$-closed subalgebra of $B(\mathcal{H})$, the algebra of bounded operators on some Hilbert space, such that the *double commutant* A'' of A is again A. (The double commutant is the space of operators which commute with the things which commute with A.) A *factor* is a von Neumann algebra A such that $Z(A) = A \cap A' = \mathbb{C} \cdot \mathrm{id}$. A *subfactor* is a unital inclusion of factors.

Why care about subfactors? Well, if you want to understand maps between von Neumann algebras, you can begin by trying to decompose your algebras into smaller pieces, and subfactors are the basic building blocks of maps.

Let S_∞ denote the *finitary symmetric group*, i.e. the group of permutations of a countable set which leave all but finitely many elements fixed. Alternatively, $S_\infty = \mathrm{colim}_n S_n$. In S_∞, cycle types are conjugacy classes, as with finite symmetric groups, and therefore there are infinitely many conjugacy classes.

Let $\mathcal{H} := \ell^2(S_\infty)$, which carries the left regular representation $\lambda\colon S_\infty \to B(\mathcal{H})$ by

$$(\lambda(g))\xi)(h) = \xi(g^{-1}h). \tag{0.2}$$

Define the *group von Neumann algebra* to be $L(S_\infty) := \mathbb{C}[\lambda(S_\infty)]''$. Taking the double commutant is called the *von Neumann closure*; $L(S_\infty)$ is a von Neumann algebra, and even a factor, though there is an argument to make here. This factor is called the *hyperfinite type* II_1 *factor*.

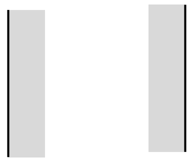
Remark 0.3. You can make this construction for any group with an infinite number of conjugacy classes (in fact, even if not, you still get a von Neumann algebra, but not a factor). For the free groups F_2 and F_3 on two, respectively three, elements, $\mathbb{C}[\lambda(F_2)] \ncong \mathbb{C}[\lambda(F_3)]$, but it is a longstanding open question whether the group von Neumann algebras are isomorphic (i.e. after von Neumann closure).

On the other hand, if G is profinite (i.e. a colimit of finite groups, as with S_∞), $L(G) \cong L(S_\infty)$. ◀

Let G be any finite subgroup of S_∞. Then we can build a subfactor: G acts on $L(S_\infty)$, and the invariants $L(S_\infty)^G \hookrightarrow L(S_\infty)$ are a subfactor.

Given a subfactor $A \subset B$, we will build a planar algebra, at least assuming A is finite-index and irreducible. Let $X := {}_AB_B$, i.e. B as an (A, B)-bimodule, with A-action implemented through the inclusion. Correspondingly, let $\overline{X} := {}_BB_A$.

Associated to this data we have some diagrammatics. We represent X by a vertical line shaded to the right, and $\overline{X}$ by a vertical line shaded to the left.

(0.4) 

X $\qquad\qquad$ $\overline{X}$

There is an evaluation map

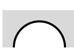
(0.5)  $: X \otimes \overline{X} = {}_AB \otimes_B B_A = {}_AB_A \longrightarrow {}_AA_A.$

In the theory of von Neumann algebras, this map is called *conditional expectation*, and has been well-studied. Correspondingly, the inclusion map ${}_AA_A \hookrightarrow {}_AB_A$ is denoted .

Two more diagrams: a map

(0.6)  $: {}_BB_B \longrightarrow \overline{X} \otimes X = {}_BB \otimes_A B_B,$

which arises from the basic construction $A \subset B \subset B \otimes_A B$; and the multiplication map

(0.7)  $: {}_BB \otimes_A B \longrightarrow {}_BB_B.$

Let $V_n \coloneqq \mathrm{End}(X \otimes \overline{X} \otimes \cdots)$, where there are n factors of X or $\overline{X}$. You can represent V_n by a box with n input and output wires.

From these diagrams we have a planar algebra. But the planar algebras which come from subfactors have extra structure. Because A and B are factors, the diagrams corresponding to the circle (two diagrams: shaded inside, or shaded outside) are constants. We adopt the convention to normalize such that both of them are the number $\mathrm{FPdim}(X)$.

There are two flavors of box, depending on whether you shade on the inside or outside; call them $V_{n,+}$ and $V_{n,-}$, depending on which of X or $\overline{X}$ is the first factor in the tensor product. These are all finite-dimensional, which follows from our assumption that $A \hookrightarrow B$ is finite index; and $\dim(V_{0,\pm}) = 1$. Moreover, we can take isotopies in S^2, rather than $\mathbb{R}^2$ — we can move strands through the point at infinity, as in Figure 1.

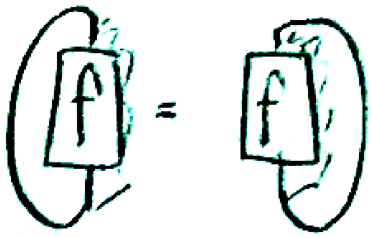


FIGURE 1. Moving a strand through the point at infinity on S^2.

FInally, we have a positivity constraint: if $a, b \in V_{n,\pm}$, $\langle a, b\rangle \coloneqq \mathrm{tr}(b^* a)$ is positive (provided $a, b \neq 0$). As before, $*$ is reflection across a horizontal line.

Call a planar algebra satisfying these three additional constraints a *subfactor planar algebra.*

Theorem 0.8 (Jones)**.** *The planar algebras coming from subfactors are subfactor planar algebras.*

Theorem 0.9 (Popa)**.** *All subfactor planar algebras arise from subfactors.*

However, both constructions are sadly non-functorial.

The subfactor planar algebra associated to a subfactor is a nice invariant of a subfactor, but is not the only interesting one. The first invariant associated to a subfactor is Jones' *index*,

$$[B : A] \coloneqq \dim_A B. \tag{0.10}$$

This is equal to $\mathrm{FPdim}(X)^2$. Actually making sense of this dimension can be a little funny, and you may have to dip into some analysis, but it can be done.

Jones proved a theorem characterizing the possible indices which made people sit up and take notice.

Theorem 0.11 (Jones [Jon83])**.** *The index is either of the form* $4\cos^2(\pi/n)$*, for* $n = 3, 4, 5, \dots,$ *or is an element of* $[4, \infty]$*, and all of these can arise.*

The indices of the form $4\cos^2(\pi/n)$ are "classical," less surprising, but the new, continuous ones are stranger.

The principal graph of a subfactor is another useful invariant — in fact, its ability to remember important information but not too much has led some people to think of it as the "Goldilocks invariant." It encodes the fusion rules of a tensor category. Consider a decomposition of $X \otimes \overline{X} \otimes X \otimes \cdots \otimes X$ into irreducible bimodules, then build a graph depicting $- \otimes X^{\pm}$. For example, if we had the rules

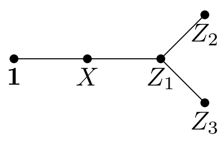
$$X \otimes \overline{X} = Z_1 \oplus \mathbf{1} \tag{0.12a}$$

$$Z_1 \otimes X = X \oplus Z_2 \oplus Z_2 \tag{0.12b}$$

$$Z_2 \otimes \overline{X} = Z_3 \otimes \overline{X} = Z_1, \tag{0.12c}$$

we would obtain the graph

(0.13)



Using this, we can build a unitary tensor category from a subfactor/subfactor planar algebra. Unitarity means that each Hom space has a positive Hermitian form, and consequently all dimensions are positive. The objects of this category are the irreducible bimodules obtained from $X \otimes \overline{X} \otimes \cdots \otimes X^{\pm}$, the morphisms are the relevant planar diagrams, and the tensor product is disjoint union of diagrams.

And this is a helpful tool for studying these tensor catgories. For small subfactor planar algebras, $\mathrm{FPdim}(X) = \sqrt{[B:A]}$ tells you about the growth rate of $\{V_{n,\pm}\}$ (assuming here that the subfactor is finite-depth or amenable). Under these assumptions, the principal graph is of ADE type, but — and a lot of work went into this — the ADE graphs that arise are D_n for n even, E_6, and E_8. One can also get extended Dynkin diagrams. These are the cases where the index is $4\cos^2(\pi/n)$ for some n, so Haagerup asked what nontrivial principal graphs can arise in the next simplest setting, where the index is above 4, but not by much. Figure 2 delineates the possible graphs. Bisch [Bis98] showed B_i doesn't work: its

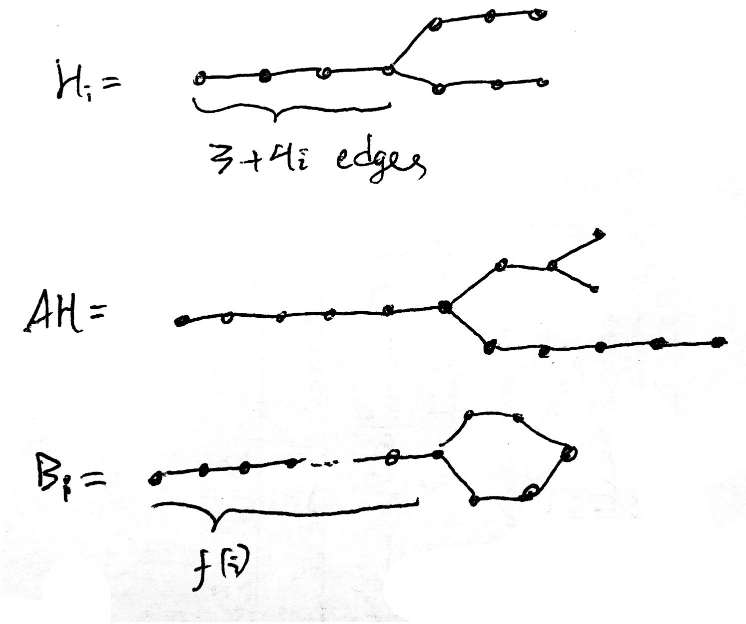


FIGURE 2. Haagerup's graphs for the case when the index is a little more than 4.

fusion rules are nonassociative. Asaeda-Haagerup [AH99] constructed subfactors realizing H_0 and AH; then Asaeda-Yasuda [AY09] showed that H_i for $i \geq 2$ cannot be realized.

This leaves one case, H_1.

Theorem 0.14 (Bigelow-Morrison-Peters-Snyder [BMPS12])**.** *There is a planar algebra realizing the graph H_1.*

This algebra is called the *extended Haagerup planar algebra.* It is a positive planar algebra generated, as a planar algebra, by $S \in V_8$ with some relations. It has Frobenius-Perron dimension about equal to 2.09218, which indicates we're just barely in the region of continuous index. The relations are described by pictures, as in Figure 3.

You can write down any random set of relations, and will probably get something zero or infinite. The work that went into Theorem 0.14 was showing that this is finite, and interesting.

References

[AH99] M. Asaeda and U. Haagerup. Exotic subfactors of finite depth with Jones indices $(5+\sqrt{13})/2$ and $(5+\sqrt{17})/2$. *Comm. Math. Phys.*, 202(1):1–63, 1999. https://arxiv.org/abs/math/9803044. 3

[AY09] Marta Asaeda and Seidai Yasuda. On Haagerup's list of potential principal graphs of subfactors. *Comm. Math. Phys.*, 286(3):1141–1157, 2009. https://arxiv.org/abs/0711.4144. 3

[Bis98] Dietmar Bisch. Principal graphs of subfactors with small Jones index. *Math. Ann.*, 311(2):223–231, 1998. 3

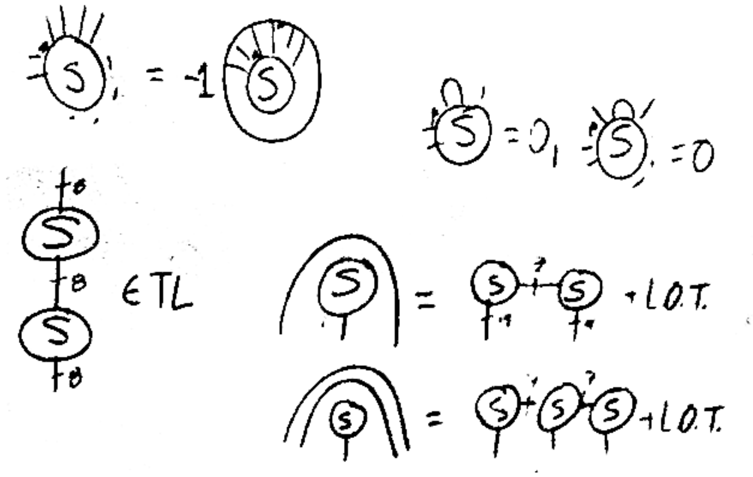


FIGURE 3. Diagrammatics for relations in the extended Haagerup planar algebra.

[BMPS12] Stephen Bigelow, Scott Morrison, Emily Peters, and Noah Snyder. Constructing the extended Haagerup planar algebra. *Acta Math.*, 209(1):29–82, 2012. https://arxiv.org/abs/0909.4099. 3

[Jon83] V. F. R. Jones. Index for subfactors. *Invent. Math.*, 72(1):1–25, 1983. 2