ZHENGHAN WANG: TOPOLOGICAL ORDERS, II

There's no single comprehensive reference for this talk, but [RW18, §4] is a good approximation.

Recall that a *topological order* is a catch-all higher-category of all universal properties of a topological phase of matter. For example, in dimension $2+1$, we have unitary modular tensor categories (though there's no proof that all topological phases in this dimension are given by modular tensor categories, nor should we necessarily expect that it's true). A *topological phase of matter* is a path-connected component of a space of topologically ordered Hamiltonians — though making this into a mathematically rigorous definition is a significant open problem.

Example 0.1 (1d Ising chain)**.** Consider a triangulation of a circle with N vertices, which we identify with $1, \dots, N$ in cyclic order. The Hilbert space L_N is a tensor product of a $\mathbb{C}^2$ for each vertex, and the Hamiltonian is

$$H_N \coloneqq -\sum_{i=1}^{N} \underbrace{\sigma_i^z \sigma_{i+1}^z}_{h_i}, \tag{0.2}$$

where $i+1$ is interpreted mod N. One can check that $[h_i, h_j] = 0$, and each h_i is a projector, simplfying analysis of the Hamiltonian. The ground state degeneracy (i.e. the dimension of the space of ground states) is 2; if $\{|0\rangle, |1\rangle\}$ is a basis for $\mathbb{C}^2$, a basis for the ground state is

$$\{|0\rangle \otimes \cdots \otimes |0\rangle, |1\rangle \otimes \cdots \otimes |1\rangle\}. \tag{0.3}$$

If you think of this as a spin model, so $|0\rangle$ is a $\uparrow$ spin and $|1\rangle$ is a $\downarrow$ spin, this is saying that the Hamiltonian enforces that neighboring spins are synchronized. ◂

Physicists worked out that there is no bosonic topological order in $(1+1)$d, at least withough additional symmetries. So why isn't the Ising model topological? The issue is that the analysis above is not stable under small perturbations of the Hamiltonian: the multiple eigenvalues that produced the ground state degeneracy split into distinct, different eigenvalues.[1] In TFT land, you have all these interesting Frobenius algebras, but they are unstable, just as the Ising model.

So let's turn to the toric code, which is actually a topological phase. Recall the Hamiltonian from last time: $H = -\sum_v A_v - \sum_P B_P$. This is a frustration-free Hamiltonian, in that $|\psi\rangle$ is a ground state iff it's fixed by all A_v and B_P operators, which is a local condition. Relatedly, the Hamiltonian is a sum of commuting projections.

The key insight is that if $A_v|\psi\rangle = |\psi\rangle$, then ψ, which is a function from the edges to $\{0,1\}$, must have an even number of 1s on the edges adjacent to a the vertex v. This implies that $S \coloneqq \psi^{-1}(1)$ is a sum of cycles. Now, $B_P|\psi\rangle$ switches the 0s and 1s on ∂P, which is the same thing as adding the boundary ∂P. Every boundary is a sum of such cycles, so what we recover is the set of functions on $H_1(M;\mathbb{Z}/2)$. Something a little weird happened here, because homology is a subquotient, and the space of ground states is a subspace, but it's OK. For example, on a torus, the ground state degeneracy is 4. There's an argument to be made as to why this is nontrivial, unlike the Ising model.

We can now upgrade the definition of a topological order to a better one, which might be quite close to whatever we eventually discover is the right definition.

Definition 0.4. An n-dimensional Hamiltonian schema is *topological* if it's gapped; its ground states form an error-correcting code; its ground states are locally indistinguishable; and the ground state degeneracy does not depend on the cellulation once the manifold is fixed.

This is due to Bravyi-Hastings-Michalakis [BHM10]. The idea of an error-correcting code is the source of the name "code" in toric code; this isn't really related to programming. In a quantum error-correcting

[1] One of the eigenvalues is $-N$. If you want energy to be positive, this is a bit perplexing, but energy is really an $\mathbb{R}$-torsor, and you can shift up by adding some multiple of the identity to the Hamiltonian, relabeling the same ground states with different energy values, so long as the energy is bounded below.

code, one has a large finite-dimensional Hilbert space of *physical qubits*, and a subspace which contains the information we'd like to protect against errors. This is sort of analogous to the Gauss-Bonnet theorem: the Euler characteristic of a surface does not change, even as you adjust the curvature. The subspace V behaves like a one-dimensional subspace, even if it isn't: when you project back down after a measurement, either you get the projection you started with, or you get zero! The toric code is an error-correcting code.

Local indistinguishability is a little tricky to write down carefully; it's expressed in terms of density matrices.

Anyways, once we have a topological order, at least in dimension $2 + 1$, we can try to extract a modular tensor category. We will focus on the ground states and also the first and second excited states, given by the eigenspaces of the first two eigenvalues above the ground state energy. These correspond to excitations that cannot be factored into pieces, and which are called *elementary excitations* — though this is not a completely mathematical term.

Let's try this for the toric code. We have two conservation laws: $\prod_v A_v = \mathrm{id}$ and $\prod_P B_P = \mathrm{id}$. This means you can't just break invariance under a single operator; you have to do two, in order to cancel signs out. So, for example, you could ask for a state $|\psi\rangle$ such that $A_v|\psi\rangle = \pm|\psi\rangle$, where the sign is $+$ except at exactly two vertices v_1 and v_2. This is an elementary excitation with energy 2 above the ground state.

These elementary excitations are anyons: they behave like particles, in that you can move them around. The toric code corresponds to the modular tensor category $D(\mathbb{Z}/2)$, with simple objects $\{1, e, m, \psi\}$ and $e \otimes m = m \otimes e = \psi$; the above excitation corresponds to producing two e particles located at v_1 and v_2. Similarly, there is a minimal excitation that creates two m particles, specified by asking for $B_P|\psi\rangle = \pm|\psi\rangle$ where the sign is $+$ except at exactly two plaquettes.

And now something weird happens. e and m are bosons, but if you braid them around each other, which corresponds to applying a sequence of operators winding their positions around each other, the ground state wavefunction is mutliplied by -1, which is strange for two bosons. This sounds physical, but can be translated into a precise mathematical statement leading you to a modular tensor category; the beginnings of this story date back to Kitaev's original paper on the toric code [Kit03].

The toric code has been well-studied, but the proof that the topological order, in the sense of Definition 0.4, for the toric code is $\mathbb{Z}/2$-Dijkgraaf-Witten theory was done only recently, in [CDH+19]. There are a few other correspondences that are open problems with a similar flavor.

- More generally, the low-energy TQFT of the Levi-Wen model should be the Turaev-Viro TQFT.
- The low-energy TQFT of the Walker-Wang model (which is in dimension $3 + 1$) should be the Crane-Yetter TQFT.
- The low-energy TQFT of the Williamson-Wang model should be the TQFT constructed by Cui from the same data.

These should all be true, though.

Another difficult problem is that of Haldane's *hardcore boson Hamiltonian* on the honeycomb lattice: this is not a commuting projector Hamiltonian, which should have the topological order of $\mathfrak{su}(2)_1$ (i.e. quantum $\mathfrak{su}_2$ at level 1). Computers can work with it and provide evidence, but nobody knows how to analytically solve it and prove that. Similarly, a model called the $J_1 - J_2$ *anti-ferromagnetic Heisenberg model* at $J_1 = J_2 = 1$ on the kagome lattice should have the same topological order as the toric code. Again, this is borne out by numerical simulations, but no proof is known. This is particularly interesting because there is an actual material called Herbertsmithite which can be coaxed into this topological order, so it would be good to better understand it.

These days, researchers are moving beyond anyons and anyon models in a few ways, such as studying symmetry defects, gapped boundaries, $(2+1)$-dimensional black holes, and fractons. *Fracton order* should have a definition similar to Definition 0.4, but the condition that the ground state degeneracy is independent of the cellulation is thrown out, replaced by the condition that it's unbounded!

Example 0.5 (Haah code [Haa11])**.** The Haah code is a $(3 + 1)$-dimensional example of fracton order, discovered by a computer search by Jeongwaan Haah. (Again, this hasn't been mathematically shown yet, but is certainly true.) There is a cubic lattice (say, on T^3), with a $\mathbb{C}^2 \otimes \mathbb{C}^2$ on each vertex. The Hamiltonian is a sum over the cubes, with operators associated to each edge of the cube.

This is an error-correcting code, and its ground state degeneracy depends on the lattice and is unbounded. It's very unclear what the continuum limit is, but if one runs renormalization group flow on this model, the degrees of freedom increase. This implies that the continuum limit cannot be a Lorentz-invariant quantum field theory; whatever it is will be interesting to discover.

Suppose the lattice on the torus has length L in all three directions. In this case the ground state degeneracy of the Haah code is $O(e^{\beta L})$ for some $\beta > 0$. What other possible growth rates are possible in fracton phases? It would be particularly interesting to find a smaller, but still unbounded, growth rate.

Another interesting question is: this only works on cubic lattices. Are there fracton phases which can be defined on more general lattices, akin to topological orders? (For example, the toric code works on any cellulation.) ◀

References

[BHM10] Sergey Bravyi, Matthew B. Hastings, and Spyridon Michalakis. Topological quantum order: Stability under local perturbations. *Journal of Mathematical Physics*, 51(9):093512, 2010. https://arxiv.org/abs/1001.0344. 1

[CDH+19] Shawn X. Cui, Dawei Ding, Xizhi Han, Geoffrey Penington, Daniel Ranard, Brandon C. Rayhaun, and Zhou Shangnan. Kitaev's quantum double model as an error correcting code. 2019. https://arxiv.org/abs/1908.02829. 2

[Haa11] Jeongwan Haah. Local stabilizer codes in three dimensions without string logical operators. *Phys. Rev. A*, 83:042330, Apr 2011. https://arxiv.org/abs/1101.1962. 2

[Kit03] A.Yu. Kitaev. Fault-tolerant quantum computation by anyons. *Annals of Physics*, 303(1):2 – 30, 2003. https://arxiv.org/abs/quant-ph/9707021. 2

[RW18] Eric C. Rowell and Zhenghan Wang. Mathematics of topological quantum computing. *Bull. Amer. Math. Soc. (N.S.)*, 55(2):183–238, 2018. https://arxiv.org/abs/1705.06206. 1