JAMES TENER: SEGAL CFTS

This will be a relatively introductory talk on the mathematics of CFTs, beginning with the definitions for functorial CFT, then passing to vertex operator algebras, chiral theories, and, near the end, the relationship to conformal nets. We will focus on two-dimensional CFTs.

We would like to model 2d CFTs as symmetric monoidal functors from a bordism category to some category of vector spaces. The objects of the bordism catgory should be compact, smooth 1-manifolds, and the morphisms (appropriate isomorphism classes of) oriented compact surfaces with a conformal structure, i.e. an equivalence class of metrics under conformal transformations. In dimension 2, this is made simpler by the fact that on closed surfaces, a conformal structure is equivalent to a complex structure. The target category will be something like topological vector spaces, maybe Hilbert spaces, maybe Fréchet spaces. The axioms for this were written down by Segal [Seg88].

However, this isn't exactly what we'll talk about in this talk. These are "full CFTs," but we'll focus on "chiral CFTs," as in Terry Gannon's talk. Confusingly, researchers on both full and chiral CFTs both just call their subjects CFTs.

Anyways, in a full CFT, let Σ be a pair of pants with a specified conformal structure; if $V \coloneqq V(S^1)$ denotes the topological vector space assigned to a circle, this assigns a map $V \otimes V \to V$. We take as an ansatz from physics that V splits as

$$V = \bigoplus_{\lambda \in \Lambda} V_\lambda \otimes \tilde{V}_\lambda, \tag{0.1}$$

where the V_λ pieces are the *chiral pieces* of V, and $\tilde{V}_\lambda$ are the *anti-chiral pieces.*

But really, we should expect a family of vector spaces associated to a circle, and many different maps between them, since we have many different conformal structures. Segal axiomatizes this as a *weak CFT*. You can reshape this data into a category of vector spaces and maps between them, and this is the genesis of the idea that you can extract a modular tensor category from a CFT; we will zoom in on the tensor unit of that tensor category, obtaining a simpler structure called a vertex operator algebra.

The definition of a vertex operator algebra (VOA) is not very enlightening, so instead we'll discuss the meaning of the axioms, which should make it easier to parse and digest the definition on Wikipedia. The point of a VOA is to axiomatize a chiral CFT in the *vaccum sector* (i.e. corresponding to the tensor unit), and restricted to conformal surfaces of genus zero. We'll give a geometrically-motivated definition of a VOA.

Definition 0.2. A *vertex operator algebra* (VOA) is data of a topological vector space V and data of, for all genus-zero n-punctured Riemann surfaces with parameterized boundary Σ, a map

$$Z_\Sigma \colon V^{\otimes(n-1)} \longrightarrow V, \tag{0.3}$$

satisfying some axioms, notably that gluing of Riemann surfaces is sent to composition of the Z_Σ maps. We also ask that this mutliplication depends holomorphically on Σ: any such Riemann surface is biholomorphic to a subset of $\mathbb{C}$, so you can imagine taking a family given by translating by w for $w \in \mathbb{C}$; then we ask that the multiplication map is a holomorphic function in w.

Unlike what you might be used to in TFT, the maps for diffeomorphic but not isomorphic surfaces are not equivalent! In fact, if you had topological invariance rather than holomorphic invariance, this notion of a VOA rapidly collapses to that of a commutative algebra — so you can think of VOAs as things like commutative algebras, but complexified: we have a complex parameter space of multiplications, which vary holomorphically.

Remark 0.4. You can describe these as holomorphic algebras for a certain operad, though that's a nontrivial theorem. ◀

If you want to understand the Wikipedia definition of a VOA, work with the two-holed annulus, and expand formally around the puncture. This gives you functions $Y(s^{L_0}-, w)r^{L_0}-$, which satisfy some axioms, and these are what appear in the dictionary definition of a VOA.

Example 0.5 (WZW model)**.** Pick a simple complex finite-dimensional Lie algebra $\mathfrak{g}$ and a positive integer k called the *level*. The level defines a certain infinite-dimensional representation V of the *loop algebra* $L\mathfrak{g} := C^\infty(S^1, \mathfrak{g})$, called the *vacuum representation* of level k. ◀

Given Σ, a pair of pants with a conformal structure, we want to define a map $Z_\Sigma \colon V \otimes V \to V$ such that for all $f \in \mathcal{O}_{\mathrm{hol.}}(\Sigma; \mathfrak{g})$,

$$f|_{\partial_{\mathrm{out}}\Sigma} \cdot Z_\Sigma(v_1 \otimes v_2) = Z_\Sigma(f|_{\partial_{\mathrm{in}}\Sigma} \cdot v_1 \otimes v_2) + Z_\Sigma(v_1 \otimes f|_{\partial_{\mathrm{out}}\Sigma} \cdot v_2). \tag{0.6}$$

Equation (0.6) is called the *Segal commutation relations*. We will eventually be able to do this up to a complex scalar; that problem is resolved using central charge.

To obtain the maps satisfying the Segal commutation relations, we will use a trick called *holomorphic induction*. The idea is that "Z_Σ satisfies the Segal relations" makes sense, and we can try to do something universal. So we can ask for a pair of $u \in V$ and Z_Σ such that all other such pairs satisfying the Segal condition factor through (u, Σ). A chiral CFT is essentially a formalization of what axioms these are supposed to satisfy — "supposed to" because there's an issue with the fact that we'd like the representation to have positive energy again, but this is not actually known to be true.

Anyways, this leads one to the definition (or a sketch of the definition) of a Segal CFT.

- We want for all C^∞ closed 1-manifolds S, a category $\mathcal{C}(S)$ equipped with a functor $\mathcal{C}(S) \to \mathcal{T}op\mathcal{V}ect$ (or maybe Fréchet or Hilbert spaces), and
- for all compact Riemann surface bordisms Σ, a functor $F_\Sigma \colon \mathcal{C}(S_{\mathrm{in}}) \to \mathcal{C}(S_{\mathrm{out}})$, and a map $Z_\Sigma \colon V_\lambda \to V_{F_\Sigma(\lambda)}$.

This is the first thing you'd write down if you wanted a generalization of topological field theory but for more than one vector space. But we need another axiom to control how this behaves in families of Σ; one way to guarantee this is via a third axiom,

- for every annulus A with parameterized boundary, an isomorphism $T_A F_A \cong \mathrm{id}$, with a positive-energy condition.

The first two axioms stitch together into a functor from a bordism category not to $\mathcal{V}ect$, but to a category of *concrete categories*, i.e. equipped with a functor to $\mathcal{T}op\mathcal{V}ect$.[1] This is called a *weak CFT*.

This structure induces a VOA on the vector space assigned to the disc. So this is quite a bit of structure, and would also encode a great deal of the associated representation theory coming from a given CFT. Thus these are quite difficult to construct.

In the last part of this lecture, we will make contact with the idea of conformal nets, a different approach to conformal field theory. We will try to go from a VOA (as defined geometrically above) to a conformal net. This forces us to restrict to unitary CFTs, so Hilbert spaces, rather than just topological vector spaces. To do this, we will need to pass through a conjecture on when we can pass from pairs of pants to all Riemann surfaces. Take an $(n+1)$-punctured genus 0 Riemann surface and identify some subset of the incoming boundary with some subset of the outgoing boundary; call such a space an *extended n-to-1 Riemann surface*.

Conjecture 0.7. Reasonable VOAs take values on extended n-to-1 Riemann surfaces, i.e. given only the data we had before, there is a canonical way to assign invariants to these spaces.

This is a step in the direction towards extended CFT (in the higher-categorical sense). You should think of the extension as unique because nonsingular surfaces are dense in the space of extended surfaces. This is sort of an analysis question, asking whether a limit exists. Recent work of the speaker attempts to use this and then build conformal nets.

Theorem 0.8 (Tener [Ten19])**.** *For all WZW models, this produces conformal nets.*

Work continues on the general setting, and on related questions.

References

[Seg88] G. B. Segal. *The Definition of Conformal Field Theory*, pages 165–171. Springer Netherlands, Dordrecht, 1988. 1

[Ten19] James E. Tener. Fusion and positivity in chiral conformal field theory. 2019. https://arxiv.org/abs/1910.08257. 2

[1] There's a slight category number mismatch here, which we're not going to worry about right now.

THANK ORGANIZERS!

Noah Emily Victor / Chelsea ~~Ezra~~ Scott

(Intro for those who like!)

MSRI Intro workshop
Jan 2020

Goals: 1) Not research
2) Introduce Segal CFT
3) Introduce VOA ideas, help read up

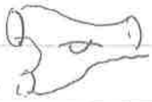
Segal CFTs

Part I: Functorial (chiral) CFT

Functorial QFT: "Bord" $\to$ "Vec"

choices / params: • geometry • dimensions / ext # • flavors of vector spaces • ext # (only? bimodules?)

2d conformal field theory (because it's interesting / accessible)



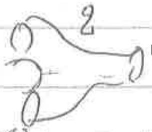
C^∞ 2 mflds w/ boundary, with conformal structure, meaning every tangent space has an inner product (metric) well-defined up to positive scaling

In fact, in 2d orientation + conformal = Complex

So maybe a (full) 2d CFT is a

2-$\mathbb{C}$-Bord { Mor: 2d $\mathbb{C}$-mfld bordisms / iso ; Ob: C^∞ 1 mflds } $\xrightarrow{\text{symmetric } \otimes\text{-functor}}$ Top Vec { Mor: cont. linear maps ; Ob: Topological Vector spaces }

add diffeos to get id.



$V(S_1) \otimes V(S_1)$ ... $V(S_1)$

Σ

$V(S_1) \otimes V(S_1) \to V(S_1)$ $\rightsquigarrow$ Correlation functions $\langle Z_\Sigma(v_1 \otimes v_2), v_3^* \rangle$

That object is not the subject of the talk.

Ansatz: In a "nice" (= rational?) CFT

$V(S^1) =: V = \bigoplus_{\lambda \in \Lambda} V_\lambda \otimes \bar{V}_\lambda$ where $\{V_\lambda\}$ encodes the part of the theory which is chiral (i.e. quantities depend holomorphically on geometry)

A chiral CFT encodes the structure enjoyed by these $\{V_\lambda\}$. Instead of a single vector space assigned to the boundary, there is a collection, and instead of a single map, you get a space of maps (conformal blocks / chiral word. fns). You would try to build the full CFT out of this data by making judicious choices ((Frobenius) algebra - see Fuchs Runkel Schweigert

categories $\to$

wake back up! Independent.

Part II VOAs, or, chiral Segal CFT in genus zero and the vacuum:
I said there were many V.S. (a category) and maps, but I am going to zoom in the unit object and genus zero. Non-categorical part.

Huang '91 a "punctured" version ≅ VOA.

Here's one definition of a "geometric VOA" (See Huang for a related one). ⚠ Lying. There is an anomaly related to central charge.

Defn:

- a topological vector space V
- for every "genus zero n-to-1 Riemann surface Σ" w/ parametrized boundary a linear map Z_Σ

$Z_\Sigma : V\otimes\cdots\otimes V \to V$

n components parametrized by S^1

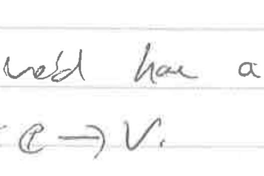
≅

s.t. · if $\Sigma \cong \Sigma'$ $Z_\Sigma = Z_{\Sigma'}$ (along parametrization)

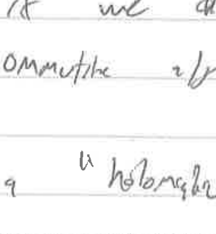
· ~~holomorphic~~ in Σ gluing ⟷ composition

· holomorphic in Σ

$Z_\Sigma \circ_i Z_{\Sigma'} = Z_{\Sigma''}$



e.g.



$Z_{\Sigma(w)}$ holomorphic in w.

Observe: if we demanded topological invariance, we'd have a commutative algebra w/ unit $(1) : \mathbb{C} \to V$.

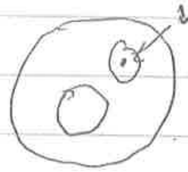
Like a "holomorphic enrichment" of the notion of algebra

Relation to Standard VOAs:

$Z\left(\text{disk}, 1 \right) = Y(\delta^{L_0} -, z)\, r^{L_0} - \; : \hat V \otimes \hat V \to \hat V$

Zooming at a pt

Moore–Segal



is a field

Operator-valued function

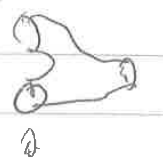
Example $\mathfrak g$ simple Lie algebra, $k \in \mathbb Z_{\geq 0}$

V_0 = vacuum rep for $\mathfrak g$ at level k

$L\mathfrak g = C^\infty(S^1, \mathfrak g_{\mathbb C})$

$\exists!$ up to scalar map $Z_\Sigma : V_0 \otimes V_0 \to V_0$

anomaly, central charge



such that $\forall f \in \mathcal O_{hol}(\Sigma; \mathfrak g_{\mathbb C})$

Segal commutation relations

$$f|_{\partial_{out}} Z_\Sigma(v_1 \otimes v_2) = Z_\Sigma(f|_{\partial_{in,1}} v_1 \otimes v_2) + Z_\Sigma(v_1 \otimes f|_{\partial_{in,2}} v_2)$$

Like a twisted version of Lie algebra intertwiner.
Like a Lie algebra, we can define modules. There are pos. energy reps V_λ $\lambda \in P^k$ (for VOA-module def, think module over algebra over operad). Note:

$V_\lambda, V_\mu, \mathcal U$ are $L\mathfrak g$-modules, can say what a map $Z_\Sigma : V_\lambda \otimes V_\mu \to \mathcal U$ satisfies Segal comm. rels.

Def The universal induction of V_λ, V_μ along Σ is the a pair $(\mathcal U, Z_\Sigma)$ s.t. any other pair factors through

In fact, we get operators $\forall\ \Sigma$. $\rightsquigarrow (u, z_\Sigma)$

Def (Sketch) A segal CFT is

"naive" QFT

-C-fixed
→ Concrete
CNS
(e ↝ Bale)

1) a) $\forall$ C^∞ 1-mfld S a linear cat $C(S)$ b) ... embedding $C(S) \longrightarrow$ Top Vec

2) a $\forall$ $\mathbb{C}$ cobordism Σ a functor $F_\Sigma : C(\partial_{in} C) \to C(\partial_{out} C)$ $\underbrace{}_{\text{compatible w/ composition}}$ b) natural maps $Z_\Sigma : V_\lambda \to F_\Sigma(V_\lambda)$

3) annuli $A \rightsquigarrow T_A : C(\partial_{in} A) \cong C(\partial_{out} A)$, positive energy, holomorphic.
Makes $C(S^1)$ into braided category.
Holomorphic induction is not known to work. Positive energy Gluing? Segal's issue.
Work in progress w/ André to build out of CNs.

Analogue

Object (///) gives VOA.

Approx: to CNS. Broader notion of geometry
Extended Graph/ Segal CFT allow $\partial_{in} \cap \partial_{out} \neq \emptyset$.

$$A(\mathcal{I}) = \left\{ \rightsquigarrow Z_\Sigma(v \otimes -) \;\middle|\; \Sigma = \text{(disk)},\ v \in V \right\}$$

"?"

Thm Works for all WZW, many others. Used to analyze rep thy, prove rationality.