Categorifications & Lie Algebra Actions on categories arising from representation theory III

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My Representation Theorists should care about 2-morphisms

Deligne Category: "Universal tensor category interpolating representations of algebraic (super) groups"

tensor category: C-linear, symmetric monoidal cat, \( \text{End}(1) = \mathbb{C} \)

ribbon: \( V^* \otimes V \xrightarrow{\omega} 1 \quad 1 \rightarrow V \otimes V^* \)

elem object \( V = \bigotimes \in \text{End}(1) = \mathbb{C} \)

Fix Sec 3

\[ \text{Rep}_0(\text{GL}_g) = \text{universal bimod cat gen by an object } V \text{ of dim } g \text{ and its dual } V^* \]
objects: finite words in \( \{ V, \Lambda \} \)
morphisms: e.g. \( \begin{array}{c}
\Lambda \\
\Lambda \\
\Lambda \\
\Lambda \\
\end{array} \)

relations: isotopies & loops evaluate to \( \delta \).

**II** \( \text{Rep} \{ \rho \} \) same as **I** except that \( V \cong V^* \)
objects, \( \mathbb{N} \)
morphs, e.g. \( \begin{array}{c}
\Lambda \\
\Lambda \\
\Lambda \\
\Lambda \\
\end{array} \) \( 4 \)

relations: isotopies & loops evaluate to \( \delta \).

**III** \( \text{Rep} \{ \rho \} \) same as **II**, but enriched in super vector spaces (Caulcember-Ehrig)\(^{(2)} \)
\( w \) deg \( V, \Lambda \) is 1 (Brundan-Ellis-\(^{(2)} \))

\[ x = n \quad x = g \quad W = - B V = - 0 \]

\[ \Rightarrow \dim V = 0 \]
\[ \text{Rep}(\cdot) := \text{Kurouban envelope of } \text{Rep}(\mathfrak{su}(n)) \]

**Representation Categories**

| I | \( \mathfrak{gl}(m|n) \rightarrow \mathbb{C}^{m|n} \) (supervector sp. of \( \text{dim} \, (m|n) \)) |

\[ \text{Lie superalgebra } \rightarrow \text{G} := \text{GL}(m|n) \text{ supergroup} \]

\[ \text{Rep}(G) := \text{fin. dim. repns of } G \]

| II | \( V = \mathbb{C}^{m|n} \) \( v = 2m \text{ or } 2m+1 \) \( w/ \text{non-degenerate bilinear form } (\cdot, \cdot) \) which is super symmetric (i.e., symmetric on even part, skew-symmetric on odd) |

\[ \text{dim } V := r-n = 2n \text{ superdimension} \]

\[ \text{osp}(m|2n) = \{ A \in \mathfrak{gl}(m|2n) | (A V, \omega) + (-1)^{\frac{|A|}{2}} (V, A \omega) = 0 \} \]

\[ \rightarrow G = \text{Osp}(m|2n) \text{ Lie supergroup } \Rightarrow \text{Rep}(G) \]
\[ V = \mathbb{C}^{n \times n} \]

\[ p(n) = \{ A \in \text{gl}(n) | A^t = -A \} \text{ but now } (\cdot, \cdot) \text{ is an odd symmetric bilinear form} \]

Up to one more family \( g(n) \) and a few exceptional cases, there are all basic simple Lie super-algebras (Kac).

In our cases:

\begin{enumerate}
\item Tensor functor \( F : \text{Rep}(\mathfrak{g}) \rightarrow \text{Rep}(G) \)
\item \( V \) of dim 8 \( \rightarrow V \) is full
\item \( \text{Generates } \text{Rep}(G) \text{ as a tensor category.} \)
\end{enumerate}

In particular, any indecomp. projective object appears as a summand in some \( V \) or \( \mathfrak{g} \).
Comes-Wilson, Brundan-S, Ehrig-S

[LB0EHILNSS]

Understanding \( V^* \otimes V \) as \( \mu \) morphs between them gives a good understanding of \( \text{Rep}(G) \) (as abelian cat).

\( \text{Rep}(G) \) is not semisimple \( \Leftrightarrow \exists Z \) and \( r, n \) chosen in a good way.

Fix \( S \in \mathbb{Z} \), which simple objects appear in a Jordan-Hölder series of a projective? \( [P(\lambda) \cdot L(\mu)] = ? \)

in \( \text{Rep}(G) \) multiplicities.

Serganova, Brundan-S

given by Kazhdan-Lusztig polynomials \( (S_n, S_i \times S_{n-i}) \) for \( n \gg 0 \)

Explicit formulas (combinatorial) [LBD......S]

Ehrig-S, Gruson-Serganova (\( B_n, A_{n-1} \)) KL polys \( n \gg 0 \)
Categorification & super vs. classical

Def. Affine Deligne Category $\text{Rep}^{\text{aff}}(O)$

exactly as $\text{Rep}(O)$ but we write one extra generator $\mathbf{1}$ for morphisms

$$\mathbf{1}$$

relations: as before plus

$$\mathbf{1} = \mathbf{1} + 1 - \mathbf{1}$$

$\text{AVW - relations}$

$\text{ABMW - relations}$

Universal property of $\text{Rep}/\text{Higher Schur-Weyl/Dueling (super) (classical)}$

$V$-nat. $G = \text{OSp}(r/2n)$-module $\text{Rep}_{g} := \text{Rep}(O)$

$\text{End}_{\text{Rep}}(d) \rightarrow \text{End}(V \otimes d)$

$\text{End}_{\text{Rep}^{\text{aff}}(d)} \rightarrow \text{End}(V \otimes d)$

$\text{Rep}^{\text{aff}} = \text{Rep}^{\text{aff}}(O)$
On the other hand, let \( M \) be a repn of \( \text{(classical) } \mathfrak{so}(N) \quad (N \gg 0) \)

\[
\text{End } (d) \rightarrow \text{End } (M \otimes (C^N)^{2d})
\]

\[
\text{Rep } \mathfrak{so}(N)
\]

\[
\text{central repn of } \mathfrak{so}(N)
\]

\[
CN \cong (CN)^* \text{ via fixed bil. form}
\]

\[
\text{swap of the factor } CN
\]

\[
i \leftrightarrow i + 1
\]

\[
\text{evaluation } \Lambda : C^n \otimes C^n \rightarrow C
\]

\[
\text{coeval. } \Psi : C \rightarrow C^n \otimes C^n
\]

\[
\text{action of } C = \mathbb{Z} \otimes X \otimes X^* \text{ on } \text{Mat}(C^N)
\]

Connect \( N \) and \( S \)

Consider for \( M \) a special module depending on \( S \)

Consider \( p \in \text{so}(N) \) of type \( A_{2M-1} \). Consider weight

\[
\text{weight} \rightarrow \tilde{w}_0 = (\frac{\delta}{2}, \ldots, \frac{\delta}{2})
\]

N even
A "universal module"

\[ M^p(S_{W_0}) = \text{maximal } p\text{-locally finite dim. quotient of } U(g) \otimes C_{S_{W_0}} \]
\[ \sigma = \mathfrak{so}(N) \]

Let \( A, d, s := \text{End} (d) \)

**Theorem (Enhry - 5):**

1. \( A, d, s / (d, d, d, d, d, d, d) \xrightarrow{\sim} \text{End} (M^p(S_{W_0}) \otimes (C \otimes V) \otimes \mathfrak{g}) \)

\[ a = \frac{s-1}{2} \quad \beta = N - \delta \]

2. Let \( e \) be the projection onto eigen spaces for \( f \) with small eigenvalues.

Then \( \text{End}_{s, s} ((M^p(S_{W_0}) \otimes (C \otimes V) \otimes \mathfrak{g}) e) \)

\[ S_{d, s} \]

The kernel of \( S_{d, s} \rightarrow \text{End} (V \otimes d) \) can be described explicitly dep. on \( r, n \)